Periodic solutions of non-linear Schrödinger equations: A para-differential approach

J.-M. Delort
Université Paris 13, Institut Galilée,
CNRS, UMR 7539, Laboratoire Analyse Géométrie et Applications
99, Avenue J.-B. Clément,
F-93430 Villetaneuse

Abstract

This paper is devoted to the construction of periodic solutions of non-linear Schrödinger equations on the torus, for a large set of frequencies. Usual proofs of such results rely on the use of Nash-Moser methods. Our approach avoids this, exploiting the possibility of reducing, through para-differential conjugation, the equation under study to an equivalent form for which periodic solutions may be constructed by a classical iteration scheme.

0 Introduction

This paper is devoted to the existence of families of periodic solutions of Hamiltonian non-linear Schrödinger equations on the torus $\mathbb{T}^d$. Our goal is to show that such results may be proved without using Nash-Moser methods, replacing them by a technically simpler conjugation idea.

We consider equations of type

$$(-i\partial_t - \Delta + \mu)u = \epsilon \frac{\partial F}{\partial \bar{u}}(\omega t, x, u, \bar{u}, \epsilon) + \epsilon f(\omega t, x)$$

where $t \in \mathbb{R}$, $x \in \mathbb{T}^d$, F is a smooth function, vanishing at order 3 at $(u, \bar{u}) = 0$, f is a smooth function on $\mathbb{R} \times \mathbb{T}^d$, 2π-periodic in time, ω a frequency parameter, μ a real parameter and $\epsilon > 0$ a small number. One wants to show that for ϵ small and ω in a Cantor set whose complement has small measure, the equation has time periodic solutions.

Let us recall known results for that type of problems. The first periodic solutions for non-linear wave or Schrödinger equations have been constructed by Kuksin [21] and Wayne [24]: they were working in one space dimension, with x staying in a compact interval, and imposing on the extremities of this interval convenient boundary conditions. Later on, Craig and Wayne [15, 16] treated the same problem for time-periodic solutions defined on $\mathbb{R} \times \mathbb{S}^1$. Periodic solutions of

This work was partially supported by the ANR project *Equa-disp*.
Keywords: Non-linear Schrödinger equations, Periodic solutions. MSC 35Q55, 35B10, 35S50.

non-linear wave equations in higher space dimensions (on $\mathbb{R} \times \mathbb{T}^d$, $d \geq 2$) have been obtained by Bourgain [9]. These results concern non-linearities which are analytic. More recently, some work has been devoted to the same problem when the non-linearity is a smooth function: Berti and Bolle [5] have proved in this setting existence of time-periodic solutions for the non-linear wave equation on $\mathbb{R} \times \mathbb{T}^d$. We refer also to the paper of Berti, Bolle and Procesi [6], where the case of equations on Zoll manifolds is treated. Very recently, Berti and Procesi [7] have studied the same problem, for wave or Schrödinger equations, on a homogeneous space. We refer also to the books of Craig [14] and of Kuksin [22] for more references.

The proofs of all above results rely on the use of the Nash-Moser theorem, to overcome unavoidable losses of derivatives coming from the small divisors appearing when inverting the linear part of the equation. Our goal here is to show that one may construct periodic solutions of non-linear Schrödinger equations (for large sets of frequencies), using just a standard iterative scheme instead of the quadratic scheme of the Nash-Moser method. This approach allows one to separate on the one hand the treatment of losses of derivatives coming from small divisors, and on the other hand the question of convergence of the sequence of approximations, while in a Nash-Moser scheme, both problems have to be treated at the same time. The basic idea is inspired by our work [17] concerning linear Schrödinger equations with smooth time dependent potential. It is shown in that paper that a linear equation of type $(i\partial_t - \Delta + V(t,x))u = 0$ may be reduced by conjugation to an equation of type $(i\partial_t - \Delta + V_{\mathrm{D}})v = Rv$, where R is a smoothing operator and V_{D} a block diagonal operator of order zero. We aim at applying a similar method when the linear potential V is replaced by a non-linear one, so that, in the reduced equation, the block-diagonal operator V_{D} depends on v itself, and R sends essentially H^s to H^{2s-a} (where a is a fixed constant, and H^s the Sobolev scale). It is pretty clear that such a reduced equation will be solvable by a standard iterative scheme, even if the inversion of $i\partial_t - \Delta + V_{\mathrm{D}}$ loses derivatives because of small divisors, since such losses are recovered by the smoothing properties of R in the right hand side.

Before describing the different sections of the paper, let us give some more references and add some comments. There are actually a few results concerning existence of periodic solutions which do not appeal to Nash-Moser theorem. Bambusi and Paleari [1, 2] constructed such solutions without making use of Nash-Moser or KAM methods, but only for a family of frequency parameters of measure zero (instead of a set of parameters whose *complement* has small measure). Related results, concerning the case of rational frequencies, may be found in chapter 5 of the book of Berti [3]. Recently, Gentile and Procesi [19] found, for analytic non-linearities, an alternative approach to Nash-Moser using expansions in terms of Lindsted series.

Let us also mention that we restrict in this paper to one of the many variants that may be considered when constructing periodic solutions. Most of the known results we cited so far concern the case of periodic solutions of the non-linear equation, whose frequency is close to the frequency of a periodic solution of the linear equation obtained for $\epsilon = 0$. The problem may be written, using a Liapounov-Schmidt decomposition, as a coupling between a non-resonant equation (the (P) equation) and a resonant one (the (Q) equation). In most works, the resonant equation is a finite dimensional equation, while (P) is infinite dimensional. One uses Nash-Moser to solve (P), getting a solution depending on finitely many parameters. Plugging this solution in (Q), one gets for these finitely many parameters an equation in closed form, that may be solved using implicit functions-like theorems. Actually, Berti-Bolle [4] have shown that such a strategy may be also adapted to the case when (Q) is completely resonant i.e. is infinite dimensional.

Since our objective here is to show that one may avoid the use of Nash-Moser theorems, we limited ourselves to the forced oscillations equation written at the beginning of the introduction, which corresponds to a (P) equation for which there is no associated (Q) equation. Note that Berti and Bolle have studied similar forced oscillations for the wave equation in [5]. It is very likely that our method could be adapted to recover as well known results for resonant periodic Schrödinger equations, even if one would have to write a detailed proof. In the same way, since the results of [17] concerning the Schrödinger equation hold not only on $\mathbb{T}^d$, but also on Zoll manifolds or on some surfaces of revolution, we conjecture that the analogue of the main theorem of this paper extends to this setting, or even to the case of a product of several Zoll manifolds.

Let us describe the organization of the paper.

The first section states the main theorem and introduces several notations.

The second section is devoted to the para-linearization of the equation. After defining convenient classes of para-differential operators, we perform a first reduction, localizing the unknown of the problem close to the characteristic variety of the linear Schrödinger operator. This is done using the standard implicit function theorem. Next, we para-linearize the equation, reducing it to

$$(-i\omega\partial_t - \Delta + V)v = R(v)v + \epsilon f$$

where V is a para-differential operator of order zero, depending on v, and $R(v)$ is a smoothing operator (Actually, we shall have to consider a system in $(v, \bar{v})$ instead of a scalar equation). A consequence of the fact that our starting equation is Hamiltonian will be that V is self-adjoint.

The third section is the heart of the paper. We construct a para-differential conjugation of the preceding equation to transform it into

$$(-i\omega\partial_t - \Delta + V_{\mathrm{D}}(w))w = R(w)w + \epsilon f$$

where $R(w)$ is still a smoothing operator, and V_{D} is block diagonal relatively to an orthogonal decomposition of $L^2(\mathbb{T}^d)$ in a sum of finite dimensional subspaces introduced by Bourgain in [12].

The fourth section is devoted to the construction of the solution to the block diagonal equation by a standard iteration scheme. We first show that on each block $-i\omega\partial_t - \Delta + V_{\mathrm{D}}(w)$ is invertible for ω outside a convenient small subset. This is done by the usual argument, exploiting that the ω-derivative of the eigenvalues of $-i\omega\partial_t - \Delta$ is large. In order that the set of excluded parameters remain small, we have to allow small divisors when inverting $-i\omega\partial_t - \Delta + V_{\mathrm{D}}(w)$. As the right hands side of the equation involves a smoothing operator $R(w)$, we may compensate the losses of derivatives coming from such small divisors, and construct a sequence of approximations of the solution.

Let us conclude this introduction with a few words concerning the limitations of our method. First, it does not seem that it could be adapted to find periodic solutions of non-linear wave equations, as the construction of section 3 relies on a specific separation property for the eigenvalues of $-\Delta$ on $\mathbb{T}^d$. On the other hand, it might be applied to equations where one has a nice separation of eigenvalues, like KdV or one dimensional water wave equations with surface tension. Second, we do not know if our method could be modified to construct quasi-periodic solutions. Recall that such solutions have been obtained for the equation set on an interval by Kuksin [21], Wayne [22], Kuksin and Pöschel [23]. The case of solutions on $\mathbb{S}^1$ has been treated by Bourgain [9]. In higher dimensions, Bourgain constructed such periodic solutions on $\mathbb{T}^2$ [11]. The case of general $\mathbb{T}^d$ has been treated by Bourgain [13] and by Eliasson and Kuksin [18]. One of the difficulties of the quasi-periodic case versus the periodic one lies in the fact that, even close to the characteristic variety, time frequencies might be much larger than space frequencies. In our proof below, the fact that these frequencies are of the same magnitude plays an important role. We do not know whether the multiscale methods of Bourgain, Eliasson, Kuksin could be combined to the arguments we use in the periodic case to construct quasi-periodic solutions without making appeal to a Newton scheme.

1 Periodic solutions of semi-linear Schrödinger equations

1.1 Statement of the main theorem

Let $\mathbb{T}^d$ $(d \geq 1)$ be the standard torus, $\mathbb{S}^1$ the unit circle. Consider a C^∞ function

$$\begin{aligned} F : (t, x, u, \bar{u}, \epsilon) &\longrightarrow F(t, x, u, \bar{u}, \epsilon) \\ \mathbb{R}\times\mathbb{T}^d \times \mathbb{C}^2 \times [0, 1] &\to \mathbb{R} \end{aligned} \tag{1.1.1}$$

which is 2π-periodic in t, and satisfies $\partial^\alpha_{u,\bar{u}} F(t, x, 0, 0, \epsilon) \equiv 0$ for $|\alpha| \leq 2$. We study the equation

$$(D_t - \Delta + \mu)u = \epsilon \frac{\partial F}{\partial \bar{u}}(\omega t, x, u, \bar{u}, \epsilon) + \epsilon f(\omega t, x) \tag{1.1.2}$$

where Δ is the Laplace operator on $\mathbb{T}^d$, $D_t = \frac{1}{i}\frac{\partial}{\partial t}$, $\epsilon \in [0, 1]$, $\mu \in \mathbb{R}$, $\omega \in \mathbb{R}^*_+$, f is a smooth function on $\mathbb{R}\times\mathbb{T}^d$, 2π-periodic in t, with values in $\mathbb{C}$, and where we look for $\frac{2\pi}{\omega}$-periodic solutions of the above equation when ϵ is small. Changing t to t/ω, we have to find solutions on $\mathbb{S}^1 \times \mathbb{T}^d$ to the equivalent equation

$$(\omega D_t - \Delta + \mu)u = \epsilon \frac{\partial F}{\partial \bar{u}}(t, x, u, \bar{u}, \epsilon) + \epsilon f(t, x) \tag{1.1.3}$$

for small enough ϵ and for ω outside a subset of small measure. To fix ideas, we shall take ω inside a fixed compact sub-interval of $]0, +\infty[$, say $\omega \in [1, 2]$.

Let us define the Sobolev space in which we shall look for solutions. If $u \in \mathcal{D}'(\mathbb{S}^1 \times \mathbb{T}^d)$, we set for $(j, n) \in \mathbb{Z} \times \mathbb{Z}^d$

$$\hat{u}(j, n) = \frac{1}{(2\pi)^{\frac{d+1}{2}}} \int_{\mathbb{S}^1\times\mathbb{T}^d} e^{-itj-in\cdot x} u(t, x)\, dtdx,$$

and define when $s \in \mathbb{R}$, $\widetilde{\mathcal{H}}^s(\mathbb{S}^1 \times \mathbb{T}^d; \mathbb{C})$ to be the space of those $u \in \mathcal{D}'(\mathbb{S}^1 \times \mathbb{T}^d)$ such that

$$\|u\|^2_{\widetilde{\mathcal{H}}^s} \stackrel{\text{def}}{=} \sum_{j\in\mathbb{Z}} \sum_{n\in\mathbb{Z}^d} (1 + |j| + |n|^2)^s |\hat{u}(j, n)|^2 < +\infty. \tag{1.1.4}$$

We shall use similar notations $\widetilde{\mathcal{H}}^s(\mathbb{S}^1 \times \mathbb{T}^d; \mathbb{C}^2)$, $\widetilde{\mathcal{H}}^s(\mathbb{S}^1 \times \mathbb{T}^d; \mathbb{R}^2)$ for $\mathbb{C}^2$ or $\mathbb{R}^2$-valued functions. Let us state our main theorem.

Theorem 1.1.1 *Let $\mu \in \mathbb{R} - \mathbb{Z}_-$. There are $s_0 > 0, \zeta > 0$ and for any $s \geq s_0$, any $q_0 > 0$, there are constants $\delta_0 \in]0,1], B > 0$ and for any $f \in \widetilde{\mathcal{H}}^{s+\zeta}(\mathbb{S}^1 \times \mathbb{T}^d; \mathbb{C})$ with $\|f\|_{\widetilde{\mathcal{H}}^{s+\zeta}} \leq q_0$, there is a subset $\mathcal{O} \subset [1,2]\times]0,1]$ such that:*

- *For any $\delta \in]0, \delta_0]$, any $\epsilon \in [0, \delta^2]$*

$$\text{meas}\{\omega \in [1,2]; (\omega, \epsilon) \in \mathcal{O}\} \leq B\delta. \tag{1.1.5}$$

- *For any $\delta \in]0, \delta_0]$, any $\epsilon \in [0, \delta^2]$, any $\omega \in [1,2]$ such that $(\omega, \epsilon) \notin \mathcal{O}$, equation (1.1.3) has a solution $u \in \widetilde{\mathcal{H}}^s(\mathbb{S}^1 \times \mathbb{T}^d; \mathbb{C})$ satisfying $\|u\|_{\widetilde{\mathcal{H}}^s} \leq B\epsilon\delta^{-1}$.*

Remark: As mentioned in the introduction, this theorem is a version, for Schrödinger equations, of theorem 1.1 of Berti-Bolle [5], which concerns wave equations. Our point will be to give a proof that does not make appeal to Nash-Moser methods.

1.2 Spaces of functions and notations

For $n \in \mathbb{Z}^d$, $u \in \mathcal{D}'(\mathbb{T}^d)$, we denote by Π_n the spectral projector

$$\Pi_n u = \hat{u}(n) \frac{e^{in\cdot x}}{(2\pi)^{d/2}} = \int_{\mathbb{T}^d} e^{-in\cdot x} u(x) \, \frac{dx}{(2\pi)^{d/2}} \frac{e^{in\cdot x}}{(2\pi)^{d/2}}. \tag{1.2.1}$$

When $u(t,x)$ is in $\mathcal{D}'(\mathbb{S}^1 \times \mathbb{T}^d)$, we use the same notation, considering t as a parameter. We shall make use of the following "separation property" result attributed by Bourgain to Granville and Spencer ([12] lemma 8.1; see also for the proof lemma 19.10 in [13]).

Lemma 1.2.1 (Bourgain) *For any $\beta \in]0, \frac{1}{10}[$, there are $\rho \in]0, \beta[$, $\theta > 0$ and a partition $(\Omega_\alpha)_{\alpha \in \mathcal{A}}$ of $\mathbb{Z}^d$ such that*

$$\begin{aligned} &\forall \alpha \in \mathcal{A}, \forall n \in \Omega_\alpha, \forall n' \in \Omega_\alpha, |n - n'| + ||n|^2 - |n'|^2| < \theta + |n|^\beta \\ &\forall \alpha, \alpha' \in \mathcal{A}, \alpha \neq \alpha', \forall n \in \Omega_\alpha, \forall n' \in \Omega_{\alpha'}, |n - n'| + ||n|^2 - |n'|^2| > |n|^\rho. \end{aligned} \tag{1.2.2}$$

For each $\alpha \in \mathcal{A}$, we choose some $n(\alpha) \in \Omega_\alpha$. There is a constant $\Theta_0 > 0$ such that, if we denote for $n \in \mathbb{Z}^d$ by $\langle n \rangle = (1 + |n|^2)^{1/2}$,

$$\Theta_0^{-1} \langle n(\alpha) \rangle \leq \langle n \rangle \leq \Theta_0 \langle n(\alpha) \rangle \tag{1.2.3}$$

for any $\alpha \in \mathcal{A}$, any $n \in \Omega_\alpha$. It also follows from (1.2.2) that, for some uniform constant $\Theta_1 > 0$,

$$\#\Omega_\alpha \leq \Theta_1 \langle n(\alpha) \rangle^{\beta d}. \tag{1.2.4}$$

For any $\alpha \in \mathcal{A}$, we set

$$\widetilde{\Pi}_\alpha = \sum_{n \in \Omega_\alpha} \Pi_n. \tag{1.2.5}$$

We define a closed subspace $\mathcal{H}^s(\mathbb{S}^1 \times \mathbb{T}^d;\mathbb{C})$ of $\widetilde{\mathcal{H}}^s(\mathbb{S}^1 \times \mathbb{T}^d;\mathbb{C})$ by

$$\begin{aligned}\mathcal{H}^s(\mathbb{S}^1 \times \mathbb{T}^d;\mathbb{C}) = \bigcap_{\alpha\in\mathcal{A}} \{u \in \widetilde{\mathcal{H}}^s(\mathbb{S}^1 \times \mathbb{T}^d;\mathbb{C}); \forall n \in \Omega_\alpha, \forall j \text{ with } |j| > K_0\langle n(\alpha)\rangle^2 \\ \text{or } |j| < K_0^{-1}\langle n(\alpha)\rangle^2,\ \hat{u}(j,n) = 0\},\end{aligned} \tag{1.2.6}$$

where $K_0 = K_0(\mu)$ will be chosen later on.

In other words, non vanishing modes (j,n) of an element u of $\mathcal{H}^s(\mathbb{S}^1 \times \mathbb{T}^d;\mathbb{C})$ have to satisfy $K_0^{-1}\langle n(\alpha)\rangle^2 \leq |j| \leq K_0\langle n(\alpha)\rangle^2$ if $n \in \Omega_\alpha$. This shows that the restriction to $\mathcal{H}^s$ of the $\widetilde{\mathcal{H}}^s$-norm given by (1.1.4) is equivalent to the square root of

$$\sum_{j\in\mathbb{Z}}\sum_{n\in\mathbb{Z}^d} \langle n\rangle^{2s}|\hat{u}(j,n)|^2 \tag{1.2.7}$$

and to the square root of

$$\sum_{\alpha\in\mathcal{A}} \langle n(\alpha)\rangle^{2s}\|\widetilde{\Pi}_\alpha u\|^2_{L^2(\mathbb{S}^1\times\mathbb{T}^d,\mathbb{C})}. \tag{1.2.8}$$

We use similar notations for spaces $\mathcal{H}^s(\mathbb{S}^1 \times \mathbb{T}^d;\mathbb{C}^2)$, $\mathcal{H}^s(\mathbb{S}^1 \times \mathbb{T}^d;\mathbb{R}^2)$,...

2 Para-linearization of the equation

The goal of this section is to rewrite (1.1.3) as a para-differential equation in the sense of Bony [8], on spaces of form (1.2.6). We first define the classes of operators we shall use.

2.1 Spaces of operators

We fix from now on some real number $\sigma_0 > \frac{d}{2} + 1$. If $s \in \mathbb{R}, q > 0$, we denote by $B_q(\mathcal{H}^s)$ the open ball with center 0, radius q in $\mathcal{H}^s(\mathbb{S}^1 \times \mathbb{T}^d;\mathbb{C})$, $\mathcal{H}^s(\mathbb{S}^1 \times \mathbb{T}^d;\mathbb{C}^2)$,...

Definition 2.1.1 *Let $m \in \mathbb{R}, q > 0, N \in \mathbb{N}, \sigma \in \mathbb{R}, \sigma \geq \sigma_0 + 2N + d + 1$. One denotes by $\Psi^m(N,\sigma,q)$ the space of maps $U \to a(U)$ defined on the open ball of center 0, radius q in $\mathcal{H}^\sigma(\mathbb{S}^1 \times \mathbb{T}^d;\mathbb{C}^2)$, with values in the space of linear maps from $C^\infty(\mathbb{S}^1 \times \mathbb{T}^d;\mathbb{C})$ to $\mathcal{D}'(\mathbb{S}^1 \times \mathbb{T}^d;\mathbb{C})$, such that, for any $n, n' \in \mathbb{Z}^d$, $U \to \Pi_n a(U)\Pi_{n'}$ is smooth with values in $\mathcal{L}(\mathcal{H}^0(\mathbb{S}^1 \times \mathbb{T}^d;\mathbb{C}))$ and satisfies for any $M \in \mathbb{N}$ with $d + 1 \leq M \leq \sigma - \sigma_0 - 2N$, any $U \in B_q(\mathcal{H}^\sigma)$, any $j \in \mathbb{N}$, any $W_1,\dots,W_j \in \mathcal{H}^\sigma(\mathbb{S}^1 \times \mathbb{T}^d;\mathbb{C}^2)$, any $n, n' \in \mathbb{Z}^d$,*

$$\begin{aligned}\left\|\Pi_n(\partial_U^j a(U)\cdot(W_1,\dots,W_j))\Pi_{n'}\right\|_{\mathcal{L}(\mathcal{H}^0)} \leq C(1+|n|+|n'|)^m\langle n-n'\rangle^{-M}\mathbb{1}_{|n-n'|\leq\frac{1}{10}(|n|+|n'|)} \\ \times\prod_{\ell=1}^j\|W_\ell\|_{\mathcal{H}^{\sigma_0+2N+M}}.\end{aligned} \tag{2.1.1}$$

Remarks:

• In (2.1.1), the decay $\langle n-n'\rangle^{-M}$ reflects the available x-smoothness of the symbol of a pseudo-differential or para-differential operator. This smoothness is controlled by the upper bound $\sigma-\sigma_0-2N$ that we assume for M. The cut-off $|n-n'|\leq\frac{1}{10}(|n|+|n'|)$ means that we are considering para-differential operators. The integer N measures some loss of smoothness, relatively to the index σ, that will appear in some expansions of operators.

• The above definition implies that if $a\in\Psi^m(N,\sigma,q)$, then $\partial_t[a(U)]$ belongs to $\Psi^m(N+1,\sigma,q)$. Actually, $\partial_t a(U)=\partial_U a(U)\cdot\partial_t U$, so (2.1.1) allows us to estimate

$$\|\Pi_n(\partial_U^j(\partial_t[a(U)])\cdot(W_1,\dots,W_j))\Pi_{n'}\|_{\mathcal{L}(\mathcal{H}^0)}$$

from $\|\partial_t U\|_{\mathcal{H}^{\sigma_0+2N+M}}\prod_{\ell=1}^j\|W_\ell\|_{\mathcal{H}^{\sigma_0+2N+M}}$, and by definition (1.2.6) of $\mathcal{H}^s$,

$$\|\partial_t U\|_{\mathcal{H}^{\sigma_0+2N+M}}\leq K_0\|U\|_{\mathcal{H}^{\sigma_0+2(N+1)+M}}\leq K_0\|U\|_{\mathcal{H}^\sigma}$$

if we assume $M\leq\sigma-2(N+1)-\sigma_0$.

The definition implies boundedness properties for the operators.

Lemma 2.1.2 *Let σ,m,N,q be as in the definition. Assume that $\sigma\geq\sigma_0+2N+d+1$. Then for any $U\in B_q(\mathcal{H}^\sigma)$, for any $s\in\mathbb{R}$, $a(U)$ is a bounded operator from $\mathcal{H}^s(\mathbb{S}^1\times\mathbb{T}^d;\mathbb{C})$ to $\mathcal{H}^{s-m}(\mathbb{S}^1\times\mathbb{T}^d;\mathbb{C})$. Moreover, $U\to a(U)$ is a smooth map from $B_q(\mathcal{H}^\sigma)$ to the space $\mathcal{L}(\mathcal{H}^s,\mathcal{H}^{s-m})$, and for any $j\in\mathbb{N}$, there is $C>0$, such that for any $U\in B_q(\mathcal{H}^\sigma)$, any $W_1,\dots,W_j\in\mathcal{H}^\sigma(\mathbb{S}^1\times\mathbb{T}^d;\mathbb{C})$*

$$\|\partial_U^j a(U)\cdot(W_1,\dots,W_j)\|_{\mathcal{L}(\mathcal{H}^s,\mathcal{H}^{s-m})}\leq C\prod_{\ell=1}^j\|W_\ell\|_{\mathcal{H}^{\sigma_0+2N+d+1}}. \tag{2.1.2}$$

Proof: One has just to apply (2.1.1) with $M=d+1$ and use that by (1.2.7), $\|v\|^2_{\mathcal{H}^s}$ is equivalent to $\sum_{n\in\mathbb{Z}^d}\langle n\rangle^{2s}\|\Pi_n v\|^2_{L^2}$. □

Let us define as well a class of smoothing operators.

Definition 2.1.3 *Let $\sigma\in\mathbb{R}$, $N\in\mathbb{N}$, $\nu\in\mathbb{N}$, with $\sigma\geq\sigma_0+2N+d+1$, $q>0$, $r\in\mathbb{R}_+$. One denotes by $\mathcal{R}^r_\nu(N,\sigma,q)$ the space of smooth maps $U\to R(U)$ defined on $B_q(\mathcal{H}^\sigma)$, with values in $\mathcal{L}(\mathcal{H}^s(\mathbb{S}^1\times\mathbb{T}^d;\mathbb{C}),\mathcal{H}^{s+r}(\mathbb{S}^1\times\mathbb{T}^d;\mathbb{C}))$ for any $s\geq\sigma_0+\nu$, such that there is for any j, any $s\geq\sigma_0+\nu$, a constant $C>0$ with*

$$\|\partial_U^j R(U)\cdot(W_1,\dots,W_j)\|_{\mathcal{L}(\mathcal{H}^s,\mathcal{H}^{s+r})}\leq C\prod_{\ell=1}^j\|W_\ell\|_{\mathcal{H}^\sigma} \tag{2.1.3}$$

for any $U\in B_q(\mathcal{H}^\sigma)$, $W_1,\dots,W_j\in\mathcal{H}^\sigma$.

Remark: Lemma 2.1.2 shows that if $r\geq0$, $\sigma\geq\sigma_0+2N+d+1$, $\Psi^{-r}(N,\sigma,q)$ is contained in $\mathcal{R}^r_0(N,\sigma,q)$.

Proposition 2.1.4 *(i) Let $\sigma \geq \sigma_0 + 2N + d + 1$, $a \in \Psi^m(N,\sigma,q)$. Then $a^* \in \Psi^m(N,\sigma,q)$.*

(ii) Let $m_1, m_2 \in \mathbb{R}$. Assume $\sigma \geq \sigma_0 + 2N + d + 1 + (m_1+m_2)_+$. Denote

$$r = \sigma - \sigma_0 - 2N - (d+1) - (m_1+m_2) \geq 0. \tag{2.1.4}$$

If $a \in \Psi^{m_1}(N,\sigma,q)$ and $b \in \Psi^{m_2}(N,\sigma,q)$, there are $c \in \Psi^{m_1+m_2}(N,\sigma,q)$ and $R \in \mathcal{R}_0^r(N,\sigma,q)$ such that

$$a(U) \circ b(U) = c(U) + R(U). \tag{2.1.5}$$

Proof: (i) follows immediately from the definition.

(ii) We define

$$c(U) = \sum_n \sum_{n'} \Pi_n[a(U)\circ b(U)]\Pi_{n'} \mathbb{1}_{|n-n'|\leq \frac{1}{10}(|n|+|n'|)}.$$

To check that (2.1.1) is satisfied by c when $j=0$ we bound

$$\|\Pi_n c(U)\Pi_{n'}\|_{\mathcal{L}(\mathcal{H}^0)} \leq \sum_k \|\Pi_n a(U)\Pi_k\|_{\mathcal{L}(\mathcal{H}^0)} \|\Pi_k b(U)\Pi_{n'}\|_{\mathcal{L}(\mathcal{H}^0)}$$

for n, n' with $|n-n'| \leq \frac{1}{10}(|n|+|n'|)$. Applying (2.1.1) to a, b with $d+1 \leq M \leq \sigma - \sigma_0 - 2N$, we get the bound

$$C(1+|n|+|n'|)^{m_1+m_2} \sum_k \langle n-k\rangle^{-M} \langle k-n'\rangle^{-M} \leq C(1+|n|+|n'|)^{m_1+m_2} \langle n-n'\rangle^{-M}.$$

One estimates $\partial_U^j c(U)$ in the same way.

The remainder $R(U) = a(U)\circ b(U) - c(U)$ will satisfy by definition of c

$$\|\Pi_n R(U)\Pi_{n'}\|_{\mathcal{L}(\mathcal{H}^0)} \leq \sum_k \|\Pi_n a(U)\Pi_k\|_{\mathcal{L}(\mathcal{H}^0)} \|\Pi_k b(U)\Pi_{n'}\|_{\mathcal{L}(\mathcal{H}^0)} \mathbb{1}_{|n-n'|>\frac{1}{10}(|n|+|n'|)}$$

so will be bounded using (2.1.1) for a, b by

$$C(1+|n|+|n'|)^{m_1+m_2} \sum_k \langle n-k\rangle^{-M} \langle k-n'\rangle^{-M} \mathbb{1}_{|k-n|\leq\frac{1}{10}(|n|+|k|)} \mathbb{1}_{|k-n'|\leq \frac{1}{10}(|n'|+|k|)} \times \mathbb{1}_{|n-n'|>\frac{1}{10}(|n|+|n'|)}$$

for any M between $d+1$ and $\sigma - \sigma_0 - 2N$. Since on the summation, either $|n-k| \geq \frac{1}{2}|n-n'|$ or $|n'-k| \geq \frac{1}{2}|n-n'|$, and $|n-n'| \leq \frac{1}{2}(|n|+|n'|)$, we get the bound

$$\|\Pi_n R(U)\Pi_{n'}\|_{\mathcal{L}(\mathcal{H}^0)} \leq C(1+|n|+|n'|)^{m_1+m_2-M} \mathbb{1}_{|n-n'|\leq\frac{1}{2}(|n|+|n'|)}$$

for any M between $d+1$ and $\sigma - \sigma_0 - 2N$. Reasoning as in the proof of lemma 2.1.2, we obtain that $R(U)$ sends $\mathcal{H}^s$ to $\mathcal{H}^{s+r}$ for any s and r given by (2.1.4). The estimates of $\partial_U^j R(U)\cdot(W_1,\dots,W_j)$ are obtained in the same way. $\square$

In the rest of this paper, we shall use several variants of the above classes. We shall denote by $\Psi^m_{\mathbb{R}}(N,\sigma,q)$ (resp. $\mathcal{R}^r_{\nu,\mathbb{R}}(N,\sigma,q)$) the subspaces of $\Psi^m(N,\sigma,q)$ (resp. $\mathcal{R}^r_\nu(N,\sigma,q)$) made of those operators $a(U)$ (resp. $R(U)$) sending real valued functions to real valued functions, i.e. satisfying $\overline{a(U)} = a(U)$ (resp. $\overline{R(U)} = R(U)$). We denote by

$$\Psi^m(N,\sigma,q)\otimes\mathcal{M}_2(\mathbb{R}),\ \mathcal{R}^r_\nu(N,\sigma,q)\otimes\mathcal{M}_2(\mathbb{R})$$

the space of 2×2 matrices with entries in $\Psi^m(N,\sigma,q)$, $\mathcal{R}^r_\nu(N,\sigma,q)$ respectively. We use similar notations for the class $\Psi^m_{\mathbb{R}}(N,\sigma,q)$, $\mathcal{R}^r_{\nu,\mathbb{R}}(N,\sigma,q)$. Finally, we shall consider operators $a(U,\omega,\epsilon)$, $R(U,\omega,\epsilon)$ depending on (ω,ϵ) staying in a bounded domain of $\mathbb{R}^2$. We shall say that these operators are C^1 in (ω,ϵ) if $(\omega,\epsilon)\to\Pi_n a(U,\omega,\epsilon)\Pi_{n'}$ (resp. $(\omega,\epsilon)\to R(U,\omega,\epsilon)$) is C^1 in (ω,ϵ) with values in $\mathcal{L}(\mathcal{H}^0)$ (resp. $\mathcal{L}(\mathcal{H}^s,\mathcal{H}^{s+r})$) and if (2.1.1) (resp. (2.1.3)) is satisfied also by $\partial_\omega a, \partial_\epsilon a$ (resp. $\partial_\omega R, \partial_\epsilon R$).

2.2 Equivalent formulation of the equation

The goal of this subsection is to reduce equation (1.1.3) to an equivalent equation for a new unknown belonging to the space $\mathcal{H}^s$ defined by (1.2.6) instead of $\widetilde{\mathcal{H}}^s$. Recall that we fixed some $\sigma_0 > \frac{d}{2}+1$.

For $\sigma\in\mathbb{R}$, we consider the space $\mathcal{H}^\sigma(\mathbb{S}^1\times\mathbb{T}^d;\mathbb{R}^2)\subset\widetilde{\mathcal{H}}^\sigma(\mathbb{S}^1\times\mathbb{T}^d;\mathbb{R}^2)$ and denote by $\mathcal{F}^\sigma(\mathbb{S}^1\times\mathbb{T}^d;\mathbb{R}^2)$ the orthogonal complement of the first space in the second one.

Definition 2.2.1 *Let $\sigma\geq\sigma_0$. Denote by $\mathcal{H}^\sigma_1,\mathcal{H}^\sigma_2$ any of the preceding spaces. Let X be an open subset of $\mathcal{H}^\sigma_1$, $k\in\mathbb{Z}$. One denotes by $\Phi^{\infty,k}(X,\mathcal{H}^{\sigma-k}_2)$ the space of C^∞ maps $G : X\to\mathcal{H}^{\sigma-k}_2$, such that for any $s\geq\sigma$, $G(u)\in\mathcal{H}^{s-k}_2$ if $u\in X\cap\mathcal{H}^s_1$ and such that:*

- *For any $s\geq\sigma$, and $u\in X\cap\mathcal{H}^s_1$, the linear map $DG(u)\in\mathcal{L}(\mathcal{H}^\sigma_1,\mathcal{H}^{\sigma-k}_2)$ extends as an element of $\mathcal{L}(\mathcal{H}^{\sigma'}_1,\mathcal{H}^{\sigma'-k}_2)$ for any $\sigma'\in[-s,s]$. Moreover, $v\to DG(v)$ is smooth from $X\cap\mathcal{H}^s_1$ to the preceding space.*

- *For any $s\geq\sigma$, any $u\in X\cap\mathcal{H}^s_1$, the bilinear map $D^2G(u)\in\mathcal{L}_2(\mathcal{H}^\sigma_1\times\mathcal{H}^\sigma_1;\mathcal{H}^{\sigma-k}_2)$ extends as an element of $\mathcal{L}_2(\mathcal{H}^{\sigma_1}_1\times\mathcal{H}^{\sigma_2}_1;\mathcal{H}^{-\sigma_3-k}_2)$ for any triple $\{\sigma_1,\sigma_2,\sigma_3\}=\{\sigma',-\sigma',\max(\sigma_0,\sigma')\}$ with $\sigma'\in[0,s]$. Moreover, $v\to D^2G(v)$ is smooth from $X\cap\mathcal{H}^s_1$ to the preceding space.*

Let us give an example of an element of $\Phi^{\infty,0}(\widetilde{\mathcal{H}}^\sigma,\widetilde{\mathcal{H}}^\sigma)$. Consider $F:\mathbb{S}^1\times\mathbb{T}^d\times\mathbb{R}^2\to\mathbb{R}^2$ a smooth function satisfying $F(t,x,0)\equiv 0, \partial_u F(t,x,0)\equiv 0$. Then, by lemma A.1 of the appendix, for $\sigma>\frac{d}{2}+1$, $u\in\widetilde{\mathcal{H}}^\sigma(\mathbb{S}^1\times\mathbb{T}^d;\mathbb{R}^2)$, $F(\cdot,u)\in\widetilde{\mathcal{H}}^\sigma(\mathbb{S}^1\times\mathbb{T}^d;\mathbb{R}^2)$ and by corollary A.2, $u\to F(\cdot,u)$ is smooth. If we define $G(u)=F(\cdot,u)$, then $DG(u)\cdot h=\partial_u F(\cdot,u)h$ which, by lemma A.3, extends as a linear map from $\widetilde{\mathcal{H}}^{\sigma'}$ to itself for any $\sigma'\in[-s,s]$, when $u\in\widetilde{\mathcal{H}}^s$ and $s>\frac{d}{2}+1$. In the same way, $D^2G(u)\cdot(h_1,h_2)=\partial^2_u F(\cdot,u)\cdot(h_1,h_2)$ extends from $\widetilde{\mathcal{H}}^{\sigma_1}\times\widetilde{\mathcal{H}}^{\sigma_2}$ to $\widetilde{\mathcal{H}}^{-\sigma_3}$ for $\sigma_1,\sigma_2,\sigma_3$ as in the statement of the definition, by lemma A.3.

Definition 2.2.2 *Let $\sigma\geq\sigma_0$, X an open subset of $\mathcal{H}^\sigma_1$, $k\in\mathbb{Z}$. One denotes by $C^{\infty,k}(X;\mathbb{R})$ the space of C^1 functions $\Phi : X\to\mathbb{R}$, such that for any $s\geq\sigma$, any $u\in X\cap\mathcal{H}^s_1$, $\nabla\Phi(u)\in\mathcal{H}^{s-k}_1$ and $u\to\nabla\Phi(u)$ belongs to $\Phi^{\infty,k}(X,\mathcal{H}^{\sigma-k}_1)$.*

If $F : \mathbb{S}^1 \times \mathbb{T}^d \times \mathbb{R}^2 \to \mathbb{R}$ is a smooth function, with $F(t,x,0) \equiv 0, \partial_u F(t,x,0) \equiv 0, \partial_u^2 F(t,x,0) \equiv 0$, and if $\Phi(u) = \int_{\mathbb{S}^1 \times \mathbb{T}^d} F(t,x,u(t,x))\, dtdx$, $\nabla\Phi(u) = \partial_u F(\cdot, u) \in \widetilde{\mathcal{H}}^s$ if $u \in \widetilde{\mathcal{H}}^s$, $s > \frac{d}{2} + 1$, (see lemma A.1) and the example following definition 2.2.1 shows that $\Phi \in C^{\infty,0}(\widetilde{\mathcal{H}}^\sigma, \mathbb{R})$ $(\sigma \geq \sigma_0)$.

Remark: In the sequel we shall have to consider elements $G(u,\omega,\epsilon)$, $\Phi(u,\omega,\epsilon)$ of the preceding spaces depending on the real parameter (ω,ϵ). We shall say that G, Φ are C^1 in (ω,ϵ) if the conditions of definition 2.2.1 (resp. definition 2.2.2) are satisfied by $G, \partial_\omega G, \partial_\epsilon G$ (resp. $\Phi, \partial_\omega \Phi, \partial_\epsilon \Phi$).

Lemma 2.2.3 *Let $\sigma \geq \sigma_0$, $k \in \mathbb{N}$, X an open subset of $\mathcal{H}_1^\sigma$, $G \in \Phi^{\infty,-k}(X, \mathcal{H}_2^{\sigma+k})$, Y an open subset of $\mathcal{H}_2^{\sigma+k}$ containing $G(X)$, $\Phi \in C^{\infty,k}(Y,\mathbb{R})$. Then $\Phi \circ G \in C^{\infty,0}(X,\mathbb{R})$.*

Proof: The assumption on G implies that for $v \in X \cap \mathcal{H}_1^s$, $s \geq \sigma$ and for σ' with $|\sigma'| \leq s$

$$DG(v) \in \mathcal{L}(\mathcal{H}_1^{\sigma'}, \mathcal{H}_2^{\sigma'+k}) \subset \mathcal{L}(\mathcal{H}_1^{\sigma'}, \mathcal{H}_2^{\sigma'}). \tag{2.2.1}$$

Moreover since $\nabla\Phi \in \Phi^{\infty,k}(Y, \mathcal{H}_2^\sigma)$, for $v \in X \cap \mathcal{H}_1^s$, $G(v) \in Y \cap \mathcal{H}_2^{s+k}$, so that $\nabla\Phi(G(v)) \in \mathcal{H}_2^s$ and for any σ'' with $|\sigma''| \leq s+k$, $(D(\nabla\Phi))(G(v))$ is in $\mathcal{L}(\mathcal{H}_2^{\sigma''}, \mathcal{H}_2^{\sigma''-k})$. In particular, for any σ' with $|\sigma'| \leq s$

$$D(\nabla\Phi)(G(v)) \in \mathcal{L}(\mathcal{H}_2^{\sigma'+k}, \mathcal{H}_2^{\sigma'}). \tag{2.2.2}$$

We deduce from (2.2.1) that $\nabla(\Phi\circ G)(v) = {}^tDG(v)\cdot(\nabla\Phi)(G(v))$ belongs to $\mathcal{H}_1^s$ when $v \in X \cap \mathcal{H}_1^s$. Let us check that $\nabla(\Phi \circ G)$ belongs to $\Phi^{\infty,0}(X, \mathcal{H}_1^\sigma)$. If $u \in X \cap \mathcal{H}_1^s$ $(s \geq \sigma)$ and $h \in \mathcal{H}_1^{\sigma'}$ with $\sigma' \in [-s, s]$, we write

$$\begin{aligned} D[\nabla(\Phi\circ G)(v)]\cdot h = {}&{}^tDG(v)\cdot((D\nabla\Phi)(G(v))\cdot DG(v)\cdot h) \\ &+ (D({}^tDG)(v)\cdot h)\cdot\nabla\Phi(G(v)). \end{aligned} \tag{2.2.3}$$

By (2.2.1), (2.2.2) the first term in the right hand side belongs to $\mathcal{H}_1^{\sigma'}$. To check that the last term in (2.2.3) belongs to the same space, we integrate it against $h' \in \mathcal{H}_1^{-\sigma'}$. We get

$$\int [(D({}^tDG)(v)\cdot h)\cdot\nabla\Phi(G(v))]h'\, dtdx = \int (\nabla\Phi)(G(v))D^2G(v)\cdot(h,h')\, dtdx. \tag{2.2.4}$$

By definition 2.2.1,

$$D^2G(v)\cdot(h,h') \in \mathcal{H}_2^{-\max(\sigma_0,\sigma')+k} \subset \mathcal{H}_2^{-\max(\sigma_0,\sigma')}.$$

Since $\nabla\Phi(G(v)) \in \mathcal{H}_2^s \subset \mathcal{H}_2^{\max(\sigma_0,\sigma')}$, this shows that the right hand side of (2.2.4) defines a continuous linear form in $h' \in \mathcal{H}_1^{-\sigma'}$.

We study now $D^2[\nabla(\Phi\circ G)(v)]\cdot(h_1,h_2)$ with $(h_1,h_2) \in \mathcal{H}_1^{\sigma_1} \times \mathcal{H}_1^{\sigma_2}$. To prove that $D^2[\nabla(\Phi \circ G)(v)]\cdot(h_1,h_2)$ belongs to $\mathcal{H}_1^{-\sigma_3}$, we compute for $h_3 \in \mathcal{H}_1^{\sigma_3}$

$$D^2\int \nabla(\Phi\circ G)(v)h_3\, dtdx = D^2\int [(\nabla\Phi)(G(v))][DG(v)\cdot h_3]\, dtdx.$$

We get the following contributions (up to symmetries) for the action on $(h_1,h_2)\in\mathcal{H}_1^{\sigma_1}\times\mathcal{H}_1^{\sigma_2}$

$$
\begin{aligned}
&\int[(\nabla\Phi)(G(v))][D^3G(v)\cdot(h_1,h_2,h_3)]\,dtdx\\
&\int[D((\nabla\Phi)(G(v)))\cdot h_1][D^2G(v)\cdot(h_2,h_3)]\,dtdx\\
&\int[(D\nabla\Phi)(G(v))\cdot D^2G(v)\cdot(h_1,h_2)][DG(v)\cdot h_3]\,dtdx\\
&\int[(D^2\nabla\Phi)(G(v))\cdot(DG(v)\cdot h_1,DG(v)\cdot h_2)][DG(v)\cdot h_3]\,dtdx.
\end{aligned}
\tag{2.2.5}
$$

On the first line in (2.2.5), we may assume for instance $h_1\in\mathcal{H}_1^{\sigma'}$, $h_2\in\mathcal{H}_1^{-\sigma'}$, $h_3\in\mathcal{H}_1^{\max(\sigma_0,\sigma')}$. Since $u\to D^2G(u)$ is C^1 on $X\cap\mathcal{H}_1^{\max(\sigma_0,\sigma')}$ with values in $\mathcal{L}_2(\mathcal{H}_1^{\sigma'}\times\mathcal{H}_1^{-\sigma'};\mathcal{H}_2^{-\max(\sigma_0,\sigma')+k})$, the second factor in the integrand belongs to $\mathcal{H}_2^{-\max(\sigma_0,\sigma')+k}$, so may be integrated against $\nabla\Phi(G(v))\in\mathcal{H}_2^s\subset\mathcal{H}_2^{\max(\sigma_0,\sigma')}$ for $s\geq\sigma'\geq0$ and $s\geq\sigma$.

On the second line of (2.2.5), $D^2G(v)\cdot(h_2,h_3)\in\mathcal{H}_2^{-\sigma_1+k}$. On the other hand $D((\nabla\Phi)(G(v)))\cdot h_1\in\mathcal{H}_2^{\sigma_1}$ by (2.2.1), (2.2.2), which allows one to integrate the product of the two factors.

On the third line of (2.2.5), $DG(v)\cdot h_3\in\mathcal{H}_2^{\sigma_3+k}$. The other factor is given by the action of $(D\nabla\Phi)(G(v))$ on $D^2G(v)\cdot(h_1,h_2)\in\mathcal{H}_2^{-\sigma_3+k}$, whence again the wanted duality in the integral, using (2.2.2).

Finally, on the last line of (2.2.5), we integrate $DG(v)\cdot h_3\in\mathcal{H}_2^{\sigma_3+k}$ against the action of $(D^2\nabla\Phi)(G(v))$ on a couple belonging to $\mathcal{H}_2^{\sigma_1+k}\times\mathcal{H}_2^{\sigma_2+k}\subset\mathcal{H}_2^{\sigma_1}\times\mathcal{H}_2^{\sigma_2}$. Since this vector is in $\mathcal{H}_2^{-\sigma_3-k}$ by definition of $C^{\infty,k}(Y,\mathbb{R})$, we get the conclusion. $\square$

Let us write an equivalent form of equation (1.1.3) using the above classes of functions. Since the Hamiltonian F in (1.1.2) is real-valued, we may write (1.1.3) as a 2×2-system

$$
\begin{aligned}
(\omega D_t-\Delta+\mu)u&=\epsilon f(t,x)+\epsilon\frac{\partial F}{\partial\bar u}(t,x,u,\bar u,\epsilon)\\
(-\omega D_t-\Delta+\mu)\bar u&=\epsilon\bar f(t,x)+\epsilon\frac{\partial F}{\partial u}(t,x,u,\bar u,\epsilon).
\end{aligned}
\tag{2.2.6}
$$

We identify $u=v_1+iv_2$ (resp. $f=f_1+if_2$) to $v=\left[\begin{smallmatrix}v_1\\v_2\end{smallmatrix}\right]$ (resp. $f=\left[\begin{smallmatrix}f_1\\f_2\end{smallmatrix}\right]$). If we set $\nabla F(v)=\begin{bmatrix}\partial F/\partial v_1\\\partial F/\partial v_2\end{bmatrix}$ and

$$
L_\omega=\begin{bmatrix}\Delta-\mu & -\omega\partial_t\\ \omega\partial_t & \Delta-\mu\end{bmatrix},
\tag{2.2.7}
$$

(2.2.6) is equivalent to

$$
L_\omega v=-\epsilon f-\epsilon\nabla_vF(t,x,v).
\tag{2.2.8}
$$

Define for $v\in\widetilde{\mathcal{H}}^s(\mathbb{S}^1\times\mathbb{T}^d;\mathbb{R}^2)$

$$
\Phi_1(v,f,\omega,\epsilon)=\frac12\int_{\mathbb{S}^1\times\mathbb{T}^d}(L_\omega v)v\,dtdx+\epsilon\int_{\mathbb{S}^1\times\mathbb{T}^d}f(t,x)v(t,x)\,dtdx
\tag{2.2.9}
$$

and

$$\Phi_2(v,\epsilon) = \int_{\mathbb{S}^1\times\mathbb{T}^d} F(t,x,v(t,x),\epsilon)\,dtdx. \tag{2.2.10}$$

Then $\nabla\Phi_1(v) = L_\omega v + \epsilon f$ so $\Phi_1 \in C^{\infty,2}(\widetilde{\mathcal{H}}^\sigma \times \widetilde{\mathcal{H}}^\sigma, \mathbb{R})$ if $\sigma \geq \sigma_0$, since by definition of $\widetilde{\mathcal{H}}^\sigma(\mathbb{S}^1 \times \mathbb{T}^d;\mathbb{R}^2)$, L_ω is bounded from $\widetilde{\mathcal{H}}^\sigma$ to $\widetilde{\mathcal{H}}^{\sigma-2}$. By the example following definition 2.2.2, $\Phi_2 \in C^{\infty,0}(\widetilde{\mathcal{H}}^\sigma,\mathbb{R})$ ($\sigma \geq \sigma_0$). Moreover equation (2.2.8) may be written

$$\nabla_v[\Phi_1(v,f,\omega,\epsilon) + \epsilon\Phi_2(v,\epsilon)] = 0. \tag{2.2.11}$$

Using the notation introduced at the beginning of this subsection, we decompose any $v \in \widetilde{\mathcal{H}}^s(\mathbb{S}^1 \times \mathbb{T}^d;\mathbb{R}^2)$ as $v = v' + v''$ on the decomposition

$$\widetilde{\mathcal{H}}^s(\mathbb{S}^1 \times \mathbb{T}^d;\mathbb{R}^2) = \mathcal{H}^s(\mathbb{S}^1 \times \mathbb{T}^d;\mathbb{R}^2) \oplus \mathcal{F}^s(\mathbb{S}^1 \times \mathbb{T}^d;\mathbb{R}^2).$$

We denote for $q > 0$ by $B_q(\widetilde{\mathcal{H}}^s)$, $B_q(\mathcal{H}^s)$, $B_q(\mathcal{F}^s)$ the ball of center 0 and radius q in these spaces. By (1.2.6), if $v \in \mathcal{F}^s(\mathbb{S}^1\times\mathbb{T}^d;\mathbb{R}^2)$, $(j,n) \in \mathbb{Z}\times\Omega_\alpha \subset \mathbb{Z}\times\mathbb{Z}^d$ and $\hat{v}(j,n) \neq 0$, then $|j| > K_0\langle n(\alpha)\rangle^2$ or $|j| < K_0^{-1}\langle n(\alpha)\rangle^2$. Moreover, since $\mu \in \mathbb{R} - \mathbb{Z}_-$, $||n|^2 + \mu| \geq c(\mu)\langle n(\alpha)\rangle^2$ when $n \in \Omega_\alpha$, for some constant $c(\mu) > 0$. If we fix K_0 large enough, and use that ω stays in $[1,2]$, we conclude that the eigenvalues of L_ω satisfy the bounds

$$|\omega j + |n|^2 + \mu| \geq c(|j| + \langle n(\alpha)\rangle^2),\ j \in \mathbb{Z}, n \in \Omega_\alpha, \alpha \in \mathcal{A}.$$

This shows that the restriction of L_ω to $\mathcal{F}^{s+2}$ is an invertible operator from $\mathcal{F}^{s+2}$ to $\mathcal{F}^s$ (uniformly in $\omega \in [1,2]$).

Let us reduce (2.2.11) to an equation on the space $\mathcal{H}^s(\mathbb{S}^1 \times \mathbb{T}^d;\mathbb{R}^2)$.

Proposition 2.2.4 *Let $\sigma \geq \sigma_0, q > 0$, $f' \in B_q(\mathcal{H}^\sigma)$. There are $\gamma_0 \in]0,1]$ and*

- *An element $(v',f'') \to \psi_2(v',f'',\omega,\epsilon)$ of $C^{\infty,0}(W_q;\mathbb{R})$ where $W_q = B_q(\mathcal{H}^\sigma(\mathbb{S}^1 \times \mathbb{T}^d;\mathbb{R}^2)) \times B_q(\mathcal{F}^\sigma(\mathbb{S}^1 \times \mathbb{T}^d;\mathbb{R}^2))$, with C^1 dependence in $(\omega,\epsilon) \in [1,2] \times [0,\gamma_0]$,*

- *An element $(v',f'') \to G(v',f'',\omega,\epsilon)$ of $\Phi^{\infty,-2}(W_q,\mathcal{F}^{\sigma+2})$, with C^1 dependence in (ω,ϵ), such that, for any given subset $A \subset [1,2] \times [0,\gamma_0]$ the following two conditions are equivalent*

(i) The function $v = (v', G(v',f'',\omega,\epsilon))$ satisfies for any $(\omega,\epsilon) \in A$

$$L_\omega v + \epsilon f + \epsilon\nabla_v\Phi_2(v,\epsilon) = 0, \tag{2.2.12}$$

where $f = f' + f''$,

(ii) The function v' satisfies for any $(\omega,\epsilon) \in A$

$$L_\omega v' + \epsilon f' + \epsilon\nabla_{v'}\psi_2(v',f'',\omega,\epsilon) = 0. \tag{2.2.13}$$

Proof: Let us write (2.2.12) as the following system

$$\begin{aligned} L_\omega v' + \epsilon f' + \epsilon\nabla_{v'}\Phi_2(v',v'',\epsilon) &= 0 \\ L_\omega v'' + \epsilon f'' + \epsilon\nabla_{v''}\Phi_2(v',v'',\epsilon) &= 0. \end{aligned} \tag{2.2.14}$$

We look for a solution of the second equation under the form $v'' = -\epsilon L_\omega^{-1} f'' + \epsilon w''$. The new unknown w'' satisfies

$$w'' = -L_\omega^{-1} \nabla_{v''} \Phi_2(v', -\epsilon L_\omega^{-1} f'' + \epsilon w'', \epsilon). \tag{2.2.15}$$

Let $q_0 > 0$ be such that for any $(v', h) \in B_q(\mathcal{H}^\sigma) \times B_q(\mathcal{F}^\sigma)$, any $\epsilon \in [0,1]$, any $\omega \in [1,2]$, $\|L_\omega^{-1}\nabla_{v''}\Phi_2(v', h, \epsilon)\|_{\mathcal{F}^{\sigma+2}} \leq q_0/2$. The fixed point theorem with parameters shows that there is $\gamma_0 \in]0,1]$ such that for any $(v', f'') \in W_q$, any $\epsilon \in [0, \gamma_0]$, equation (2.2.15) has a unique solution $w'' \in B_{q_0}(\mathcal{F}^{\sigma+2})$. We denote this solution by $G(v', f'', \omega, \epsilon)$. This is a smooth function of $(v', f'') \in W_q$, with C^1 dependence in (ω, ϵ). If moreover $(v', f'') \in \widetilde{\mathcal{H}}^s$ for some $s \geq \sigma$, it follows from (2.2.15) that $w'' \in \mathcal{F}^{s+2}$ (using that L_ω^{-1} gains two derivatives in the $\mathcal{F}^s$ scale). Let us show that G belongs to $\Phi^{\infty,-2}(W_q, \mathcal{F}^{\sigma+2})$. By definition of G

$$\begin{aligned} D_{v'}G(v', f'', \omega, \epsilon) &= -L_\omega^{-1}(\mathrm{Id} - \epsilon M''(v', f'', \omega, \epsilon)L_\omega^{-1})^{-1} M'(v', f'', \omega, \epsilon) \\ D_{f''}G(v', f'', \omega, \epsilon) &= \epsilon L_\omega^{-1}(\mathrm{Id} - \epsilon M''(v', f'', \omega, \epsilon)L_\omega^{-1})^{-1} M''(v', f'', \omega, \epsilon)L_\omega^{-1} \end{aligned} \tag{2.2.16}$$

with

$$\begin{aligned} M'(v', f'', \omega, \epsilon) &= (D_{v'}\nabla_{v''}\Phi_2)(v', -\epsilon L_\omega^{-1} f'' + \epsilon G, \epsilon) \\ M''(v', f'', \omega, \epsilon) &= -(D_{v''}\nabla_{v''}\Phi_2)(v', -\epsilon L_\omega^{-1} f'' + \epsilon G, \epsilon). \end{aligned} \tag{2.2.17}$$

Since $\Phi_2 \in C^{\infty,0}(W_q, \mathbb{R})$, when $(v', f'') \in W_q \cap \widetilde{\mathcal{H}}^s$ for some $s \geq \sigma$, $M''(v', f'', \omega, \epsilon)$ (resp. $M'(v', f'', \omega, \epsilon)$) extends as an element of $\mathcal{L}(\mathcal{F}^{\sigma'}, \mathcal{F}^{\sigma'})$ (resp. $\mathcal{L}(\mathcal{H}^{\sigma'}, \mathcal{F}^{\sigma'})$) for any $\sigma' \in [-s, s]$. We choose γ_0 small enough so that for $\epsilon \in [0, \gamma_0]$, $\epsilon\|M''(v', f'', \omega, \epsilon)L_\omega^{-1}\|_{\mathcal{L}(\mathcal{F}^\sigma, \mathcal{F}^\sigma)}$ is smaller than $1/2$. Let us check that G satisfies the first condition in definition 2.1.1. We may write the first equation in (2.2.16) as

$$D_{v'}G(v', f'', \omega, \epsilon) = -\sum_{k=0}^{2N-1} L_\omega^{-1}(\epsilon M'' L_\omega^{-1})^k M' - L_\omega^{-1}(\epsilon M'' L_\omega^{-1})^N (\mathrm{Id} - \epsilon M'' L_\omega^{-1})^{-1}(\epsilon M'' L_\omega^{-1})^N M' \tag{2.2.18}$$

and a similar formula for $D_{f''}G$. If N is chosen large enough relatively to s, and $\sigma' \in [-s, s]$, $(\epsilon M'' L_\omega^{-1})^N M'$ sends $\mathcal{H}^{\sigma'}$ to $\mathcal{F}^\sigma$, over which $(\mathrm{Id} - \epsilon M'' L_\omega^{-1})^{-1}$ is bounded. Consequently, the last contribution in (2.2.18) is in $\mathcal{F}^{s+2} \subset \mathcal{F}^{\sigma'+2}$. The sum in the right hand side being bounded from $\mathcal{H}^{\sigma'}$ to $\mathcal{F}^{\sigma'+2}$ for any $\sigma' \in [-s, s]$, we get the same property for $D_{v'}G$. We argue in the same way for $D_{f''}G$. To check the second condition in definition 2.1.1, we compute from (2.2.16), for $(h_1, h_2) \in \mathcal{H}^{\sigma_1} \times \mathcal{H}^{\sigma_2}$

$$\begin{aligned} D_{v'}^2 G(v', f'', \omega, \epsilon) \cdot (h_1, h_2) = &-L_\omega^{-1}(\mathrm{Id} - \epsilon M'' L_\omega^{-1})^{-1}[(D_{v'}M' \cdot h_1) \cdot h_2] \\ &- L_\omega^{-1}(\mathrm{Id} - \epsilon M'' L_\omega^{-1})^{-1}(\epsilon D_{v'} M'' L_\omega^{-1} \cdot h_1)(\mathrm{Id} - \epsilon M'' L_\omega^{-1})^{-1} M' \cdot h_2. \end{aligned}$$

If $\{\sigma_1, \sigma_2, \sigma_3\} = \{\sigma', -\sigma', \max(\sigma_0, \sigma')\}$, the assumption on Φ_2 implies that $D_{v'}M'$ (resp. $D_{v'}M''$) sends $\mathcal{H}^{\sigma_1} \times \mathcal{H}^{\sigma_2}$ (resp. $\mathcal{H}^{\sigma_1} \times \mathcal{F}^{\sigma_2}$) to $\mathcal{F}^{-\sigma_3}$. Using expansions as in (2.2.18), we conclude that if $(h_1, h_2) \in \mathcal{H}^{\sigma_1} \times \mathcal{H}^{\sigma_2}$, $D_{v'}^2 G(v', f'', \omega, \epsilon) \cdot (h_1, h_2) \in \mathcal{F}^{-\sigma_3+2}$. One studies in the same way $D_{v'}D_{f''}G, D_{f''}^2 G$. Since smoothness of DG, D^2G in $(v', f'') \in W_q \cap \widetilde{\mathcal{H}}^s$, as well as C^1 dependence in (ω, ϵ) are clear, we conclude that $G \in \Phi^{\infty,-2}(W_q, \mathcal{F}^{\sigma+2})$.

Let us obtain the equivalent form (2.2.13) of equation (2.2.12) or (2.2.11). By (2.2.9), (2.2.10)

$$\begin{aligned}\Phi_1(v',v'',\omega,\epsilon)+\epsilon\Phi_2(v',v'',\epsilon)=\frac{1}{2}\int(L_\omega v')v'\,dtdx+\epsilon\int f'v'\,dtdx\\+\frac{1}{2}\int(L_\omega v'')v''\,dtdx+\epsilon\int f''v''\,dtdx+\epsilon\Phi_2(v',v'',\epsilon).\end{aligned}$$

We plug in this expression the solution of the second equation in (2.2.14), namely we set $v''=-\epsilon L_\omega^{-1}f''+\epsilon G(v',f'',\omega,\epsilon)$. We get after simplification the function

$$\begin{aligned}\Psi(v',f'',\omega,\epsilon)=\frac{1}{2}\int(L_\omega v')v'\,dtdx+\epsilon\int f'v'\,dtdx\\-\frac{\epsilon^2}{2}\int(L_\omega^{-1}f'')f''\,dtdx+\epsilon\psi_2(v',f'',\omega,\epsilon)\end{aligned}$$

where

$$\psi_2(v',f'',\omega,\epsilon)=\frac{\epsilon}{2}\int G(L_\omega G)\,dtdx+\Phi_2(v',-\epsilon L_\omega^{-1}f''+\epsilon G,\epsilon). \tag{2.2.19}$$

Note that the integral in (2.2.19) is the composition of the function defined on $\mathcal{F}^\sigma$ by $w''\to\int w''(L_\omega w'')\,dtdx$, which is an element of $C^{\infty,2}(\mathcal{F}^\sigma,\mathbb{R})$, with the map

$$\begin{aligned}(v',f'')&\to G(v',f'',\omega,\epsilon)\\ \widetilde{\mathcal{H}}^\sigma&\to\mathcal{F}^{\sigma+2}\end{aligned}$$

which is an element of $\Phi^{\infty,-2}(W_q,\mathcal{F}^{\sigma+2})$. By lemma 2.2.3, we conclude that $\psi_2\in C^{\infty,0}(W_q,\mathbb{R})$.

Since G is defined as the critical point (up to an affine change of variables) of $v''\to(\Phi_1+\epsilon\Phi_2)(v',v'',\omega,\epsilon)$, and since Ψ is the corresponding critical value, we see that v' solves the first equation (2.2.14) if and only of $\nabla_{v'}\Psi(v',f'',\omega,\epsilon)=0$. This gives equation (2.2.13). □

We finish this subsection with a lemma that will be useful in the sequel. Let X be an open subset of $\mathcal{H}^{\sigma_0}(\mathbb{S}^1\times\mathbb{T}^d;\mathbb{R}^2)$, ψ an element of $C^{\infty,0}(X;\mathbb{R})$. For $v\in X\cap\mathcal{H}^{+\infty}$, $w_1,w_2\in\mathcal{H}^{+\infty}$, we set

$$L(v;w_1,w_2)=D^2\psi(v)\cdot(w_1,w_2). \tag{2.2.20}$$

This is a continuous bilinear form in $(w_1,w_2)\in\mathcal{H}^0\times\mathcal{H}^0$, by the definition of $C^{\infty,0}(X;\mathbb{R})$. By Riesz theorem, we write it

$$L(v;w_1,w_2)=\int_{\mathbb{S}^1\times\mathbb{T}^d}(W(v)w_1)w_2\,dtdx$$

for some symmetric $\mathcal{H}^0$-bounded operator $W(v)$. Since definition 2.2.2 implies that $v\to D^2\psi(v)$ is a smooth map on X with values in the space of continuous bilinear forms on $\mathcal{H}^0\times\mathcal{H}^0$, we know that $v\to W(v)$ is smooth with values in $\mathcal{L}(\mathcal{H}^0,\mathcal{H}^0)$. Consequently, we may write for $j=1,\dots,d$

$$\begin{aligned}L(v;\partial_{x_j}w_1,w_2)+L(v;w_1,\partial_{x_j}w_2)&=-\int_{\mathbb{S}^1\times\mathbb{T}^d}((\partial_{x_j}W(v))w_1)w_2\,dtdx\\&=-(\partial_vL)(v;w_1,w_2)\cdot(\partial_{x_j}v)\end{aligned} \tag{2.2.21}$$

for any $v \in X \cap \mathcal{H}^{+\infty}$, $w_1, w_2 \in \mathcal{H}^{+\infty}$.

We denote by $\mathbb{C}[X_\alpha; \alpha \in \mathbb{N}^d]$ the space of polynomials in indeterminates X_α, indexed by elements of $\mathbb{N}^d$. If $X_{\alpha_1}^{k_1} \cdots X_{\alpha_\ell}^{k_\ell}$ is a monomial, its weight will be defined as $k_1|\alpha_1| + \cdots + k_\ell|\alpha_\ell|$. The weight of any polynomial is then defined in the natural way.

Lemma 2.2.5 *For any $N \in \mathbb{N}$, any $\ell \in \mathbb{N}$, there is a polynomial $Q_N^\ell \in \mathbb{C}[X_\alpha; \alpha \in \mathbb{N}^d]$, of weight less or equal to N, and for any $q > 0$ a constant $C > 0$ such that, for any $v \in B_q(\mathcal{H}^{\sigma_0}) \cap \mathcal{H}^{+\infty} \cap X$, any $h_1, \dots, h_\ell$ in $\mathcal{H}^{+\infty}$, any $n, n' \in \mathbb{Z}^d$*

$$\begin{aligned}&\|\Pi_n \partial_v^\ell W(v) \cdot (h_1, \dots, h_\ell) \Pi_{n'}\|_{\mathcal{L}(\mathcal{H}^0)} \leq \\ &\qquad C\langle n - n'\rangle^{-N} \sum_{N_0 + \cdots + N_\ell = N} Q_{N_0}^\ell((\|\partial^\alpha v\|_{\mathcal{H}^{\sigma_0}})_\alpha) \prod_{\ell'=1}^{\ell} \|h_{\ell'}\|_{\mathcal{H}^{\sigma_0 + N_{\ell'}}}.\end{aligned} \tag{2.2.22}$$

Proof: Since ${}^t\Pi_n = \Pi_{-n}$, we may write for any $w_1, w_2 \in \mathcal{H}^{+\infty}$

$$\begin{aligned}(n_j - n_j') \int (\Pi_n W(v) \Pi_{n'} w_1) w_2 \, dt dx &= (n_j - n_j') L(v; \Pi_{n'} w_1, \Pi_{-n} w_2) \\ &= i[L(v; \partial_{x_j} \Pi_{n'} w_1, \Pi_{-n} w_2) + L(v; \Pi_{n'} w_1, \partial_{x_j} \Pi_{-n} w_2)] \\ &= -i(\partial_v L)(v; \Pi_{n'} w_1, \Pi_{-n} w_2) \cdot (\partial_{x_j} v)\end{aligned}$$

by (2.2.21). Iterating the computation, we get for

$$\langle n - n'\rangle^N \Big| \int (\Pi_n W(v) \Pi_{n'} w_1) w_2 \, dt dx \Big|$$

an estimate in terms of quantities

$$|(\partial_v^p L)(v; \Pi_{n'} w_1, \Pi_{-n} w_2) \cdot (\partial^{\alpha_1} v, \dots, \partial^{\alpha_p} v)|$$

with $|\alpha_1| + \cdots + |\alpha_p| \leq N$. By the properties of L, this is bounded from above by

$$C \|\Pi_{n'} w_1\|_{L^2} \|\Pi_{-n} w_2\|_{L^2} \prod_{p'=1}^{p} \|\partial^{\alpha_{p'}} v\|_{\mathcal{H}^{\sigma_0}}$$

when v stays in a fixed $\mathcal{H}^{\sigma_0}$-ball. This implies (2.2.22) for $\ell = 0$. The proof for general ℓ is similar, up to notations. □

2.3 Reduction to a para-differential equation

We want to construct, under the conditions of the statement of theorem 1.1.1, periodic solutions to equation (2.2.6). We have rewritten this equation under the real form (2.2.8) (or (2.2.11)). By Proposition 2.2.4, if we find a periodic solution v' for (2.2.13), we get a periodic solution v for (2.2.12), which is a rewriting of (2.2.11). We are thus reduced to finding a solution

$v' \in \mathcal{H}^\sigma(\mathbb{S}^1 \times \mathbb{T}^d; \mathbb{R}^2)$ to (2.2.13). Since the force term $f = f' + f''$ will be fixed, we no longer write the f'' dependence in the function ψ_2 defined in proposition 2.2.4. Moreover, since, in the rest of the paper, we will study only the equivalent formulation (2.2.13) of our initial problem, we drop the primes i.e. we study

$$L_\omega v + \epsilon f + \epsilon \nabla_v \psi_2(v, \omega, \epsilon) = 0 \tag{2.3.1}$$

where $v \in B_q(\mathcal{H}^\sigma(\mathbb{S}^1 \times \mathbb{T}^d; \mathbb{R}^2))$, $f \in \mathcal{H}^s(\mathbb{S}^1 \times \mathbb{T}^d; \mathbb{R}^2)$, ψ_2 is in $C^{\infty,0}(B_q(\mathcal{H}^\sigma), \mathbb{R})$ for some $\sigma \in [\sigma_0, s]$, $q > 0$ and for $\epsilon \in [0, \gamma_0]$, with $\gamma_0 \in]0, 1]$ small enough. We shall use the equivalent norms (1.2.7) and (1.2.8) on the spaces we consider.

Our objective in this subsection is to rewrite the non-linearity in (2.3.1) using para-differential operators.

Proposition 2.3.1 *Let $q > 0$, $\sigma \geq \sigma_0 + d + 1$ be given. Denote*

$$r = \sigma - \sigma_0 - d - 1. \tag{2.3.2}$$

There is an element $\widetilde{V} \in \Psi^0_{\mathbb{R}}(0, \sigma, q) \otimes \mathcal{M}_2(\mathbb{R})$, symmetric, and an element $\widetilde{R} \in \mathcal{R}^r_{0,\mathbb{R}}(0, \sigma, q) \otimes \mathcal{M}_2(\mathbb{R})$, with C^1 dependence in (ω, ϵ), such that, for any $v \in B_q(\mathcal{H}^\sigma)$, any $\epsilon \in [0, \gamma_0]$, $\omega \in [1, 2]$

$$\nabla_v \psi_2(v, \omega, \epsilon) = \widetilde{V}(v, \omega, \epsilon)v + \widetilde{R}(v, \omega, \epsilon)v. \tag{2.3.3}$$

Let us comment about the interest of the above decomposition of $\nabla_v \psi_2$. It allows us to express the non-linearity in (2.3.1) as the sum of a remainder and of the action of the para-differential potential $\widetilde{V}(v, \omega, \epsilon)$ on v. In that way, the main contribution to the non-linearity is expressed in terms of a class of operators enjoying a nice calculus. This will be exploited below to perform a block diagonalization.

We introduce some notations for the proof. For $p \in \mathbb{N}$, $v \in \mathcal{H}^0(\mathbb{S}^1 \times \mathbb{T}^d; \mathbb{R}^2)$, we set

$$\begin{aligned} \Delta_p v &= \sum_{\substack{n \in \mathbb{Z}^d \\ 2^{p-1} \leq |n| < 2^p}} \Pi_n v, \ p \geq 1, \ \Delta_0 v = \Pi_0 v \\ S_p v &= \sum_{p'=0}^{p-1} \Delta_{p'} v = \sum_{\substack{n \in \mathbb{Z}^d \\ |n| < 2^{p-1}}} \Pi_n v, \ p \geq 1, \ S_0 v = 0. \end{aligned} \tag{2.3.4}$$

We consider also the frequency cut-offs defined for $n, n' \in \mathbb{Z}^d$ by

$$S(n, n') = \sum_{|n''| \leq 2(1 + \min(|n|, |n'|))} \Pi_{n''}. \tag{2.3.5}$$

Lemma 2.3.2 *Let $\sigma \geq \sigma_0 + d + 1$, $q > 0$. There is a map $(v, \omega, \epsilon) \to W(v, \omega, \epsilon)$ defined for $v \in B_q(\mathcal{H}^\sigma)$, $\epsilon \in [0, \gamma_0]$, $\omega \in [1, 2]$, with values in the space of bounded symmetric operators on $\mathcal{H}^0(\mathbb{S}^1 \times \mathbb{T}^d; \mathbb{R}^2)$, which is C^∞ in v and has C^1 dependence in (ω, ϵ), such that for any (v, ω, ϵ)*

$$\psi_2(v, \omega, \epsilon) = \int_{\mathbb{S}^1 \times \mathbb{T}^d} [W(v, \omega, \epsilon)v]v \, dtdx \tag{2.3.6}$$

and such that the following estimate holds: There are for $(\ell, N) \in \mathbb{N}\times\mathbb{N}$ polynomials $Q_N^\ell \in \mathbb{C}[X_\alpha;\alpha\in\mathbb{N}]$, of weight less or equal to N, and there is for any $M\in\mathbb{N}$, any $\ell\in\mathbb{N}$, a constant C, depending only on ℓ, q, M, such that for any $v\in B_q(\mathcal{H}^\sigma)$, any $\epsilon\in[0,\gamma_0]$, any $\omega\in[1,2]$, any $(a_0,a_1)\in\mathbb{N}^2$, $a_0+a_1\le 1$, any $(h_1,\dots,h_\ell)\in(\mathcal{H}^\sigma)^\ell$, any $n,n'\in\mathbb{Z}^d$

$$\begin{aligned}&\|\Pi_n\partial_\omega^{a_0}\partial_\epsilon^{a_1}D_v^\ell W(v,\omega,\epsilon)\cdot(h_1,\dots,h_\ell)\Pi_{n'}\|_{\mathcal{L}(\mathcal{H}^0)}\\ &\le C\langle n-n'\rangle^{-M}\sum_{N_0+\cdots+N_\ell=M}Q_{N_0}^\ell((\|\partial^\alpha S(n,n')v\|_{\mathcal{H}^{\sigma_0}})_\alpha)\prod_{\ell'=1}^{\ell}\|S(n,n')h_{\ell'}\|_{\mathcal{H}^{\sigma_0+N_{\ell'}}}.\end{aligned}\tag{2.3.7}$$

Proof: We do not write ω,ϵ which play the role of parameters. Since ψ_2 vanishes at order 3 at $v=0$, and $S_pv\to v$ in $\mathcal{H}^\sigma$ when $p\to+\infty$, we write

$$\psi_2(v)=\sum_{p_1=0}^{+\infty}(\psi_2(S_{p_1+1}v)-\psi_2(S_{p_1}v))=\sum_{p_1=0}^{+\infty}\int_0^1(\partial\psi_2)(S_{p_1}v+\tau_1\Delta_{p_1}v)\,d\tau_1\cdot\Delta_{p_1}v.$$

Repeating the process, we get

$$\psi_2(v)=\sum_{p_1=0}^{+\infty}\sum_{p_2=0}^{+\infty}\int_0^1\int_0^1(\partial^2\psi_2)(\Omega_{p_1,p_2}(\tau_1,\tau_2)v)\,d\tau_2\cdot(\Delta_{p_2}(S_{p_1}+\tau_1\Delta_{p_1})v,\Delta_{p_1}v)\,d\tau_1$$

where $\Omega_{p_1,p_2}(\tau_1,\tau_2)=\prod_{\ell=1}^2(S_{p_\ell}+\tau_\ell\Delta_{p_\ell})$. By the discussion before lemma 2.2.5, there is a symmetric operator $\widetilde{W}(v)$ satisfying (2.2.22), such that

$$\partial^2\psi_2(v)\cdot(w_1,w_2)=\int[\widetilde{W}(v)w_1]w_2\,dtdx.$$

We set

$$\begin{aligned}W(v)=&\frac12\sum_{p_1}\sum_{p_2}\int_0^1\int_0^1\Delta_{p_1}[\widetilde{W}(\Omega_{p_1,p_2}(\tau_1,\tau_2)v)\Delta_{p_2}(S_{p_1}+\tau_1\Delta_{p_1})]\,d\tau_1d\tau_2\\ &+\frac12\sum_{p_1}\sum_{p_2}\int_0^1\int_0^1\Delta_{p_2}(S_{p_1}+\tau_1\Delta_{p_1})[\widetilde{W}(\Omega_{p_1,p_2}(\tau_1,\tau_2)v)\Delta_{p_1}]\,d\tau_1d\tau_2.\end{aligned}\tag{2.3.8}$$

This is a symmetric operator. We apply (2.2.22) to $\widetilde{W}$. Because of the cut-offs in the argument of $\widetilde{W}$ in (2.3.8), we may write $\Pi_nW(v)\Pi_{n'}=\Pi_nW(S(n,n')v)\Pi_{n'}$. Consequently, (2.2.22) implies (2.3.7). Note that since $\sigma\ge\sigma_0+d+1$, we may take some integer $M>d$, such that $\sigma_0+M\le\sigma$, so that for $v,h_{\ell'}$ in $\mathcal{H}^\sigma$, the right hand side of (2.3.7) is bounded from above by $C\langle n-n'\rangle^{-M}$. This shows that $W(v)$ is indeed bounded on $\mathcal{H}^0$. □

Proof of Proposition 2.3.1: Let h_1 be in $\mathcal{H}^{+\infty}(\mathbb{S}^1\times\mathbb{T}^d;\mathbb{R}^2)$ and write

$$D\psi_2(v,\omega,\epsilon)\cdot h_1=2\int_{\mathbb{S}^1\times\mathbb{T}^d}(W(v,\omega,\epsilon)v)h_1\,dtdx+\int_{\mathbb{S}^1\times\mathbb{T}^d}((DW(v,\omega,\epsilon)\cdot h_1)v)v\,dtdx.\tag{2.3.9}$$

Define

$$\widetilde{V}=2\sum_{n,n'}\mathbb{1}_{|n-n'|\le\frac{1}{10}(|n|+|n'|)}\Pi_nW(v,\omega,\epsilon)\Pi_{n'}.$$

Bounding in (2.3.7) $\|\partial^\alpha S(n,n')v\|_{\mathcal{H}^{\sigma_0}}$ by $C\|v\|_{\mathcal{H}^\sigma}$ when $|\alpha| \leq M \leq \sigma - \sigma_0$ and controlling $\|S(n,n')h_{\ell'}\|_{\mathcal{H}^{\sigma_0+N_{\ell'}}}$ by $C\|h_{\ell'}\|_{\mathcal{H}^{\sigma_0+M}}$, we obtain that $\widetilde{V}$ satisfies estimates (2.1.1) i.e. is an element of $\Psi^0(0,\sigma,q)$. Let us show that the remaining terms in (2.3.9) give contributions to the last term in (2.3.3). Set

$$R_1(v,\omega,\epsilon) = 2\sum_n \sum_{n'} \Pi_n W(v,\omega,\epsilon)\Pi_{n'} \mathbb{1}_{|n-n'|>\frac{1}{10}(|n|+|n'|)}.$$

We estimate

$$\left\|\Pi_n \partial_\omega^{a_0}\partial_\epsilon^{a_1}\partial_v^\ell R_1(v,\omega,\epsilon)\cdot(h_1,\dots,h_\ell)\Pi_{n'}\right\|_{\mathcal{L}(\mathcal{H}^0)} \tag{2.3.10}$$

using (2.3.7) with $M > \sigma - \sigma_0$. Since $\|S(n,n')w\|_{\mathcal{H}^{\sigma_0+\beta}} \leq C(1+\inf(|n|,|n'|))^{(\beta+\sigma_0-\sigma)_+}\|w\|_{\mathcal{H}^\sigma}$, we get for (2.3.10) the upper bound

$$C(1+|n|+|n'|)^{-M}(1+\inf(|n|,|n'|))^{M+\sigma_0-\sigma}\prod_{\ell'=1}^{\ell}\|h_{\ell'}\|_{\mathcal{H}^\sigma}$$

Taking M large enough, we deduce from that the boundedness of $R_1(v,\omega,\epsilon)$ and of its derivatives from $\mathcal{H}^s$ to $\mathcal{H}^{s+(\sigma-\sigma_0-d-1)}$ for any $s \geq \sigma_0$ i.e. $R_1 \in \mathcal{R}^r_{0,\mathbb{R}}(0,\sigma,q)$.

We treat next the last contribution to (2.3.9), defining an operator $R_2(v,\omega,\epsilon)$ by

$$\int [(DW(v,\omega,\epsilon)\cdot h)v]w\,dtdx = \int [R_2(v,\omega,\epsilon)w]h\,dtdx \tag{2.3.11}$$

for any $h, w \in \mathcal{H}^{+\infty}$. In the left hand side, we decompose the last v as $\sum_{n'}\Pi_{n'}v$ and w as $\sum_n \Pi_n w$. We bound the modulus of (2.3.11) by

$$\sum_n\sum_{n'}\|\Pi_n DW(v,\omega,\epsilon)\cdot h\Pi_{n'}\|_{\mathcal{L}(\mathcal{H}^0)}\|\Pi_{n'}v\|_{\mathcal{H}^0}\|\Pi_n w\|_{\mathcal{H}^0}. \tag{2.3.12}$$

To show that $R_2(v,\omega,\epsilon)$ is bounded from $\mathcal{H}^s$ to $\mathcal{H}^{s+r}$, we bound $\|\Pi_n w\|_{\mathcal{H}^0} \leq c_n\langle n\rangle^{-s}\|w\|_{\mathcal{H}^s}$, for a ℓ^2-sequence $(c_n)_n$ and take $h \in \mathcal{H}^{-s-r}$. We use (2.3.7) with $\ell = 1$. We bound

$$Q^1_{N_0}((\|\partial^\alpha S(n,n')v\|_{\mathcal{H}^{\sigma_0}})_\alpha)\|S(n,n')h\|_{\mathcal{H}^{\sigma_0+N_1}} \leq C(1+\inf(|n|,|n'|))^{M+s+r+\sigma_0}\|h\|_{\mathcal{H}^{-s-r}}$$

since v is bounded in $\mathcal{H}^\sigma$. Consequently, the general term of (2.3.12) is smaller than

$$C\langle n-n'\rangle^{-M}(1+\inf(|n|,|n'|))^{M+s+r+\sigma_0}\langle n\rangle^{-s}c_n\|w\|_{\mathcal{H}^s}\|h\|_{\mathcal{H}^{-s-r}}\langle n'\rangle^{-\sigma}c'_{n'}\|v\|_{\mathcal{H}^\sigma} \tag{2.3.13}$$

for some ℓ^2-sequence $(c'_{n'})_{n'}$. Taking $M = d+1$, and using the value (2.3.2) of r and $s \geq 0, \sigma \geq 0$, one checks that the sum in n, n' of (2.3.13) converges. This shows the boundedness of $R_2(v,\omega,\epsilon)$ from $\mathcal{H}^s$ to $\mathcal{H}^{s+r}$. One treats in the same way $\partial_\omega^{a_0}\partial_\epsilon^{a_1}\partial_v^\ell R_2(v,\omega,\epsilon)$. Consequently $R_2 \in \mathcal{R}^r_{0,\mathbb{R}}(0,\sigma,q)$. This concludes the proof of the proposition. □

Let us conclude this section writing the equation we are interested in in complex coordinates. By proposition 2.3.1, equation (2.3.1) may be written

$$L_\omega v + \epsilon f + \epsilon\widetilde{V}(v,\omega,\epsilon)v + \epsilon\widetilde{R}(v,\omega,\epsilon)v = 0. \tag{2.3.14}$$

We write $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^2$ and set $u = v_1 + iv_2$, $U = \begin{bmatrix} u \\ \bar{u} \end{bmatrix}$, $I' = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

Corollary 2.3.3 *Let $q > 0$, $\sigma \geq \sigma_0 + d + 1$, r given by (2.3.2). There is an element $V(U, \omega, \epsilon)$ in $\Psi^0(0, \sigma, q) \otimes \mathcal{M}_2(\mathbb{R})$ with $V(U, \omega, \epsilon)^* = V(U, \omega, \epsilon)$, there is $R(U, \omega, \epsilon)$ in $\mathcal{R}_0^r(0, \sigma, q) \otimes \mathcal{M}_2(\mathbb{R})$ such that equation (2.3.14) is equivalent to*

$$[(\omega I' D_t + (-\Delta + \mu)I) + \epsilon V(U, \omega, \epsilon)]U = \epsilon R(U, \omega, \epsilon)U + \epsilon f \tag{2.3.15}$$

(where, abusing notations, we set f for $\left[\begin{smallmatrix} f_1+if_2 \\ f_1-if_2 \end{smallmatrix}\right]$).

Proof: Write $\widetilde{V}(v, \omega, \epsilon) = (\widetilde{V}_{i,j}(v, \omega, \epsilon))_{1 \leq i,j \leq 2}$, $\widetilde{R}(v, \omega, \epsilon) = (\widetilde{R}_{i,j}(v, \omega, \epsilon))_{1 \leq i,j \leq 2}$ and note that (2.3.14) implies

$$\begin{aligned}(\omega D_t - \Delta + \mu)u = \epsilon(f_1 + if_2) - \epsilon V_{11}(U, \omega, \epsilon)u - \epsilon V_{12}(U, \omega, \epsilon)\bar{u} \\ + \epsilon R_{11}(U, \omega, \epsilon)u + \epsilon R_{12}(U, \omega, \epsilon)\bar{u}\end{aligned} \tag{2.3.16}$$

if we set

$$\begin{aligned}V_{11} &= -\frac{1}{2}[\widetilde{V}_{11} + \widetilde{V}_{22} + i(\widetilde{V}_{21} - \widetilde{V}_{12})] \\ V_{12} &= -\frac{1}{2}[\widetilde{V}_{11} - \widetilde{V}_{22} + i(\widetilde{V}_{21} + \widetilde{V}_{12})] \\ R_{11} &= \frac{1}{2}[\widetilde{R}_{11} + \widetilde{R}_{22} + i(\widetilde{R}_{21} - \widetilde{R}_{12})] \\ R_{12} &= \frac{1}{2}[\widetilde{R}_{11} - \widetilde{R}_{22} + i(\widetilde{R}_{21} + \widetilde{R}_{12})].\end{aligned} \tag{2.3.17}$$

We define $V_{21} = \overline{V}_{12}, V_{22} = \overline{V}_{11}, R_{21} = \overline{R}_{12}, R_{22} = \overline{R}_{11}$, $V = (V_{ij})_{1 \leq i,j \leq 2}$, $R = (R_{ij})_{1 \leq i,j \leq 2}$. Since ${}^t\widetilde{V} = \widetilde{V}$ and $\overline{\widetilde{V}} = \widetilde{V}$, we see that $V^* = V$ and (2.3.16), (2.3.17) imply (2.3.15). This concludes the proof. $\square$

3 Diagonalization of the problem

The goal of this section is to deduce from equation (2.3.15) a new equation where, up to remainders, $V(U, \omega, \epsilon)$ will be replaced by a block diagonal operator relatively to the decomposition $\mathcal{H}^0 = \bigoplus_\alpha \operatorname{Range}(\widetilde{\Pi}_\alpha)$ coming from (1.2.5). This is the key point, that will allow us to avoid using Nash-Moser methods in the construction of the solution performed in section 4.

3.1 Spaces of diagonal and non diagonal operators

Definition 3.1.1 *Let $\sigma \in \mathbb{R}$, $N \in \mathbb{N}$, $\sigma \geq \sigma_0 + d + 1 + 2N$, $m \in \mathbb{R}$, $q > 0$.*

(i) One denotes by $\Sigma^m(N, \sigma, q)$ the space $\Psi^m(N, \sigma, q) \otimes \mathcal{M}_2(\mathbb{R})$. Abusing notations, we also write $\mathcal{R}_\nu^r(N, \sigma, q)$ for $\mathcal{R}_\nu^r(N, \sigma, q) \otimes \mathcal{M}_2(\mathbb{R})$.

(ii) One denotes by $\Sigma^m_{\mathrm{D}}(N,\sigma,q)$ the subspace of $\Sigma^m(N,\sigma,q)$ made of those elements $A(U,\omega,\epsilon) = (A_{ij}(U,\omega,\epsilon))_{1\le i,j\le 2}$ such that $A_{12} = A_{21} = 0$ and for any $\alpha,\alpha' \in \mathcal{A}$ with $\alpha \neq \alpha'$

$$\widetilde{\Pi}_\alpha A_{11}(U,\omega,\epsilon)\widetilde{\Pi}_{\alpha'} \equiv 0, \widetilde{\Pi}_\alpha A_{22}(U,\omega,\epsilon)\widetilde{\Pi}_{\alpha'} \equiv 0. \tag{3.1.1}$$

(iii) One denotes by $\Sigma^m_{\mathrm{ND}}(N,\sigma,q)$ the subspace of $\Sigma^m(N,\sigma,q)$ made of those elements $A(U,\omega,\epsilon)$ such that for any $\alpha \in \mathcal{A}$

$$\widetilde{\Pi}_\alpha A_{11}(U,\omega,\epsilon)\widetilde{\Pi}_{\alpha} \equiv 0, \widetilde{\Pi}_\alpha A_{22}(U,\omega,\epsilon)\widetilde{\Pi}_{\alpha} \equiv 0. \tag{3.1.2}$$

Clearly, we get a direct sum decomposition $\Sigma^m(N,\sigma,q) = \Sigma^m_{\mathrm{D}}(N,\sigma,q) \oplus \Sigma^m_{\mathrm{ND}}(N,\sigma,q)$.

Definition 3.1.2 *Let $\rho \in]0,1]$. One denotes by $\mathcal{L}^m_\rho(N,\sigma,q)$ (resp. $\mathcal{L}'^m_\rho(N,\sigma,q)$) the subspace of $\Sigma^{m-\rho}(N,\sigma,q)$ given by those $A(U,\omega,\epsilon) = (A_{ij}(U,\omega,\epsilon))_{1\le i,j\le 2}$ satisfying*

$$A_{11}, A_{22} \in \Psi^{m-\rho}(N,\sigma,q),\ A_{12}, A_{21} \in \Psi^{m-2}(N,\sigma,q) \tag{3.1.3}$$

(resp. satisfying (3.1.3) and

$$A^*_{11} = -A_{11}, A^*_{22} = -A_{22}, A^*_{12} = A_{21}). \tag{3.1.4}$$

Remark: It follows from the definition and from proposition 2.1.4 (ii) that if $A \in \mathcal{L}^{m_1}_\rho(N,\sigma,q)$, $B \in \mathcal{L}^{m_2}_\rho(N,\sigma,q)$ with $\sigma \ge \sigma_0 + 2N + d + 1 + (m_1 + m_2 - 2\rho)_+$, AB is the sum of an element of $\mathcal{L}^{m_1+m_2-\rho}_\rho(N,\sigma,q)$ and of an element of $\mathcal{R}^r_0(N,\sigma,q)$ with

$$r = \sigma - \sigma_0 - (d+1) - m_1 - m_2 + 2\rho - 2N.$$

Proposition 3.1.3 *Let $A(U,\omega,\epsilon)$ be a self-adjoint element of $\Sigma^m_{\mathrm{ND}}(N,\sigma,q)$. There are an element $B(U,\omega,\epsilon)$ of $\mathcal{L}'^m_\rho(N,\sigma,q)$ and an element $R(U,\omega,\epsilon)$ of $\mathcal{R}^{r(\sigma,N)-m}_0(N,\sigma,q)$ with $r(\sigma,N) = \rho(\sigma - \sigma_0 - 2N - d - 1)$, such that*

$$B(U,\omega,\epsilon)^*(\Delta - \mu) + (\Delta - \mu)B(U,\omega,\epsilon) = A(U,\omega,\epsilon) + R(U,\omega,\epsilon) \tag{3.1.5}$$

(where ρ is given by lemma 1.2.1, for a given $\beta \in]0, \frac{1}{10}[$). Moreover $[\Delta, B]$ is in $\Sigma^m(N,\sigma,q)$.

Proof: By assumption, we may write $A(U,\omega,\epsilon) = \begin{bmatrix} a(U,\omega,\epsilon) & b(U,\omega,\epsilon) \\ b(U,\omega,\epsilon)^* & c(U,\omega,\epsilon) \end{bmatrix}$ with $a^* = a$, $c^* = c$, $\widetilde{\Pi}_\alpha a \widetilde{\Pi}_{\alpha'} = 0$, $\widetilde{\Pi}_\alpha c \widetilde{\Pi}_{\alpha'} = 0$ if $\alpha,\alpha' \in \mathcal{A}$, $\alpha \neq \alpha'$. Decompose $a = a' + a''$ with

$$a' = \sum_{n,n'} \mathbb{1}_{|n-n'| \le c(|n|+|n'|)^\rho} \Pi_n a \Pi_{n'},\ a'' = \sum_{n,n'} \mathbb{1}_{|n-n'| > c(|n|+|n'|)^\rho} \Pi_n a \Pi_{n'}$$

where c is a small positive constant. Applying (2.1.1) with $M = \sigma - \sigma_0 - 2N - d - 1$, we get

$$\begin{aligned} \|\Pi_n \partial^j_U a''(U)(W_1,\dots,W_j)\Pi_{n'}\|_{\mathcal{L}(\mathcal{H}^0)} &\le C(1+|n|+|n'|)^{m-r(\sigma,N)} \langle n-n' \rangle^{-d-1} \\ &\quad \times \mathbb{1}_{|n-n'| \le \frac{1}{10}(|n|+|n'|)} \prod_{\ell=1}^{j} \|W_\ell\|_{\mathcal{H}^\sigma} \end{aligned}$$

which implies a bound of type (2.1.3) for any $s \geq \sigma_0$, with r replaced by $r(\sigma, N) - m$. Consequently, a'' gives a contribution to R in (3.1.5) and, changing notations, we may assume that $a = a'$. We do the same for the c-contribution, so that we reduce ourselves to a, c verifying

$$\Pi_n a \Pi_{n'} = 0, \Pi_n c \Pi_{n'} = 0, \text{ if } |n - n'| > c(|n| + |n'|)^\rho. \tag{3.1.6}$$

We look for $B(U, \omega, \epsilon) = \begin{bmatrix} a_1(U, \omega, \epsilon) & b_1(U, \omega, \epsilon) \\ b_1(U, \omega, \epsilon)^* & c_1(U, \omega, \epsilon) \end{bmatrix}$ for some a_1, b_1, c_1 satisfying $a_1^* = -a_1, c_1^* = -c_1$ such that $A(U, \omega, \epsilon)$ equals the left hand side of (3.1.5). The latter may be written

$$\begin{bmatrix} [\Delta, a_1] & (\Delta - \mu) b_1 + b_1(\Delta - \mu) \\ b_1^*(\Delta - \mu) + (\Delta - \mu) b_1^* & [\Delta, c_1] \end{bmatrix}. \tag{3.1.7}$$

Consequently, we have to solve the equations

$$[\Delta, a_1] = a, (\Delta - \mu) b_1 + b_1(\Delta - \mu) = b, [\Delta, c_1] = c. \tag{3.1.8}$$

The first equation in (3.1.8) is equivalent to

$$(|n'|^2 - |n|^2) \Pi_n a_1 \Pi_{n'} = \Pi_n a \Pi_{n'} \text{ for any } n, n' \in \mathbb{Z}^d. \tag{3.1.9}$$

Since $A \in \Sigma^m_{\mathrm{ND}}(N, \sigma, q)$, (ii) of definition 3.1.1 implies that the right hand side in (3.1.9) vanishes if n, n' belong to a same Ω_α of the partition of lemma 1.2.1. Consequently, we may define

$$a_1(U, \omega, \epsilon) = \sum_{\substack{\alpha, \alpha' \in \mathcal{A} \\ \alpha \neq \alpha'}} \sum_{n \in \Omega_\alpha} \sum_{n' \in \Omega_{,\alpha'}} (|n'|^2 - |n|^2)^{-1} \Pi_n a(U, \omega, \epsilon) \Pi_{n'}. \tag{3.1.10}$$

If we use the second lower bound in (1.2.2), definition 2.1.1 and (3.1.6) with a small enough $c > 0$, we see that a_1 satisfies (2.1.1) with m replaced by $m - \rho$. Thus $a_1 \in \Psi^{m-\rho}(N, \sigma, q)$, and by (3.1.10) and the fact that $a^* = a$, we get $a_1^* = -a_1$. The last equation (3.1.8) is solved in the same way.

We are left with finding $b_1(U, \omega, \epsilon)$. The equation giving it is equivalent to

$$-(|n|^2 + |n'|^2 + 2\mu) \Pi_n b_1 \Pi_{n'} = \Pi_n b \Pi_{n'}. \tag{3.1.11}$$

Since by assumption $\mu \notin \mathbb{Z}_-$, we may always define b_1 by division. Coming back to definition 2.1.1, we see that we get an element of $\Psi^{m-2}(N, \sigma, q)$, which is moreover self-adjoint. This concludes the proof since (3.1.7) shows that by construction $[\Delta, a_1], [\Delta, c_1]$ belong to $\Psi^m(N, \sigma, q)$, and since $\Delta b_1, b_1 \Delta$ and their adjoints are in $\Psi^m(N, \sigma, q)$. □

3.2 Diagonalization theorem

The main result of this subsection is the following one, which gives a reduction for the left hand side of equation (2.3.15).

Proposition 3.2.1 *Let r be a given positive number and fix an integer N such that $(N+1)\rho \geq r+2$. Let $\sigma \in \mathbb{R}$ satisfy*

$$\sigma \geq \sigma_0 + 2(N+1) + d + 1 + r/\rho. \tag{3.2.1}$$

Let $q > 0$ be given. One may find elements $Q_j(U,\omega,\epsilon)$ in $\mathcal{L}_\rho^{-j\rho}(j,\sigma,q)$, $0 \leq j \leq N$, elements $V_{\mathrm{D},j}(U,\omega,\epsilon)$ in $\Sigma_{\mathrm{D}}^{-j\rho}(j,\sigma,q)$, $0 \leq j \leq N-1$, an element $R_1(U,\omega,\epsilon)$ in $\mathcal{R}_2^r(N+1,\sigma,q)$, with C^1 dependence in (ω,ϵ), such that if one denotes

$$Q(U,\omega,\epsilon) = \sum_{j=0}^{N} Q_j(U,\omega,\epsilon),\ V_{\mathrm{D}}(U,\omega,\epsilon) = \sum_{j=0}^{N-1} V_{\mathrm{D},j}(U,\omega,\epsilon),\ I' = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{3.2.2}$$

one gets for any $U \in B_q(\mathcal{H}^\sigma(\mathbb{S}^1 \times \mathbb{T}^d; \mathbb{C}^2))$

$$\begin{aligned}(\mathrm{Id} + \epsilon Q(U,\omega,\epsilon))^*(\omega I' D_t + (-\Delta+\mu)I + \epsilon V(U,\omega,\epsilon))(\mathrm{Id} + \epsilon Q(U,\omega,\epsilon))\\ = \omega I' D_t + (-\Delta+\mu)I + \epsilon V_{\mathrm{D}}(U,\omega,\epsilon) - \epsilon R_1(U,\omega,\epsilon).\end{aligned} \tag{3.2.3}$$

We shall prove proposition 3.2.1 constructing recursively Q_j, $0 \leq j \leq N$ so that Q_j may be written $Q_j = Q_j' + Q_j''$ with

$$\begin{aligned}&Q_j' \in \mathcal{L}_\rho'^{-j\rho}(j,\sigma,q),\ [\Delta, Q_j'] \in \Sigma^{-j\rho}(j,\sigma,q),\ j = 0,\dots,N\\ &Q_j'' \in \mathcal{L}_\rho^{-(j+1)\rho}(j,\sigma,q),\ [\Delta, Q_j''] \in \Sigma^{-(j+1)\rho}(j,\sigma,q),\ j = 0,\dots,N-1\\ &Q_N'' = 0.\end{aligned} \tag{3.2.4}$$

We compute first the left hand side of (3.2.3).

Proposition 3.2.2 *Let r,σ,N satisfying $(N+1)\rho \geq r+2$ and $\sigma \geq \sigma_0 + 2(N+1) + d + 1 + r$. Let $Q(U,\omega,\epsilon) = \sum_{j=0}^{N} Q_j(U,\omega,\epsilon)$ be given, with $Q_j = Q_j' + Q_j''$ satisfying (3.2.4). There are elements*

$$S_j(U,\omega,\epsilon) \in \mathcal{L}_\rho^{-(j+1)\rho}(j,\sigma,q),\ j = 0,\dots,N-1 \text{ with } [\Delta, S_j] \in \Sigma^{-(j+1)\rho}(j,\sigma,q) \tag{3.2.5}$$

where S_j depends only on Q_ℓ', $0 \leq \ell \leq j$, Q_ℓ'', $0 \leq \ell \leq j-1$;
There are elements

$$V_j(U,\omega,\epsilon) \in \Sigma^{-j\rho}(j,\sigma,q),\ 0 \leq j \leq N$$

with $(V_j)^* = V_j$, V_j depending only on $Q_\ell, \ell \leq j-1$;
There is an element $R \in \mathcal{R}_2^r(N+1,\sigma,q)$ such that, if we set

$$\begin{gathered}V^N(U,\omega,\epsilon) = \sum_{j=0}^{N} V_j(U,\omega,\epsilon),\ S^N(U,\omega,\epsilon) = \sum_{j=0}^{N-1} S_j(U,\omega,\epsilon),\\ Q' = \sum_{j=0}^{N} Q_j',\ Q'' = \sum_{j=0}^{N} Q_j'',\ \widetilde{L}_\omega = \omega I' D_t + (-\Delta+\mu)I,\end{gathered}$$

the following equality holds

$$\begin{aligned}(\mathrm{Id} + \epsilon Q)^*[\widetilde{L}_\omega + \epsilon V](\mathrm{Id} + \epsilon Q) = \widetilde{L}_\omega + \epsilon V^N + \epsilon[(S^N)^* \widetilde{L}_\omega + \widetilde{L}_\omega (S^N)]\\ + \epsilon[Q'^*(-\Delta+\mu) + (-\Delta+\mu)Q']\\ + \epsilon[Q''^* \widetilde{L}_\omega + \widetilde{L}_\omega Q''] + \epsilon R.\end{aligned} \tag{3.2.6}$$

Before starting the proof, we compute some commutators.

Lemma 3.2.3 *(i) One may find $A_j \in \Sigma^{-j\rho}(j-1,\sigma,q)$, $1 \leq j \leq N$, A_j depending only on Q_ℓ, $\ell \leq j-1$ and satisfying $A_j^* = A_j$, one may find $B_j \in \mathcal{L}_\rho^{-(j+1)\rho}(j,\sigma,q)$, $0 \leq j \leq N-1$, B_j depending only on Q'_ℓ, $\ell \leq j$ and Q''_ℓ, $\ell \leq j-1$ and satisfying $[\Delta, B_j] \in \Sigma^{-(j+1)\rho}(j,\sigma,q)$, one may find $R \in \mathcal{R}_2^r(N+1,\sigma,q)$, such that, if one sets $A = \sum_{j=1}^N A_j$, $B = \sum_{j=0}^{N-1} B_j$,*

$$[Q^*, \widetilde{L}_\omega]Q + Q^*[\widetilde{L}_\omega, Q] = A + B^*\widetilde{L}_\omega + \widetilde{L}_\omega B + R. \tag{3.2.7}$$

(ii) One may find A_j as above for $1 \leq j \leq N$, $B_j \in \mathcal{L}_\rho^{-(j+1)\rho}(j,\sigma,q)$, $0 \leq j \leq N-1$, satisfying $[\Delta, B_j] \in \Sigma^{-(j+1)\rho}(j,\sigma,q)$, B_j depending only on Q'_ℓ, $\ell \leq j$, Q''_ℓ, $\ell \leq j-1$, and $R \in \mathcal{R}_2^r(N+1,\sigma,q)$ such that, with the same notations as in (i),

$$Q^*\widetilde{L}_\omega Q = A + B^*\widetilde{L}_\omega + \widetilde{L}_\omega B + R. \tag{3.2.8}$$

Proof: (i) Let us write

$$\begin{aligned}
[\widetilde{L}_\omega, Q] &= -[\Delta, Q] + \omega[I'D_t, Q]\\
&= -[\Delta, Q] + \omega I'[D_t, Q] + \omega[I', Q]D_t\\
&= -[\Delta, Q] + \omega I'[D_t, Q] + [I', Q]I'(\Delta-\mu) + [I', Q]I'\widetilde{L}_\omega.
\end{aligned}$$

The left hand side of (3.2.7) may be written

$$\begin{aligned}
&-Q^*[\Delta, Q] + \omega Q^*I'[D_t, Q] + Q^*[I', Q]I'(\Delta-\mu)\\
&-[Q^*, \Delta]Q + \omega[Q^*, D_t]I'Q + (\Delta-\mu)I'[Q^*, I']Q\\
&\qquad\qquad + Q^*[I', Q]I'\widetilde{L}_\omega + \widetilde{L}_\omega I'[Q^*, I']Q.
\end{aligned} \tag{3.2.9}$$

Denote by $\tilde{A}$ the sum of the first two lines in (3.2.9). Then $\tilde{A}$ is self-adjoint and may be written as $\sum_{j=1}^{2N+2} \tilde{A}_j$, where $\tilde{A}_j$ is the sum of the following terms

$$\sum_{\substack{j_1+j_2=j-1\\ 0\leq j_1,j_2\leq N}} (-[Q^*_{j_1}, \Delta]Q_{j_2} - Q^*_{j_2}[\Delta, Q_{j_1}])\ (j \geq 1), \tag{3.2.10}$$

$$\omega \sum_{\substack{j_1+j_2=j-2\\ 0\leq j_1,j_2\leq N}} (Q^*_{j_1}I'[D_t, Q_{j_2}] + [Q^*_{j_2}, D_t]I'Q_{j_1}))\ (j \geq 2), \tag{3.2.11}$$

$$\sum_{\substack{j_1+j_2=j-1\\ 0\leq j_1,j_2\leq N}} (Q^*_{j_1}[I', Q_{j_2}]I'(\Delta-\mu) + (\Delta-\mu)I'[Q^*_{j_2}, I']Q_{j_1})\ (j \geq 1). \tag{3.2.12}$$

Let us check that we may write $\tilde{A}_j = A_j + R_{1,j}$ with A_j in $\Sigma^{-j\rho}(\min(N+1, j-1), \sigma, q)$ and $R_{1,j}$ in $\mathcal{R}_0^r(\min(N+1, j-1), \sigma, q)$. Since $\mathcal{L}_\rho^{-j_\ell\rho}(j_\ell, \sigma, q) \subset \Sigma^{-(j_\ell+1)\rho}(j_\ell, \sigma, q)$, it follows from

(3.2.4) and from (ii) of proposition 2.1.4 that the general term in (3.2.10) may be written as a contribution to A_j plus a remainder belonging to $\mathcal{R}_0^{r_1}(\min(N,j-1),\sigma,q)$ with

$$r_1 = \sigma - \sigma_0 - 2N - (d+1) + (j_1+j_2+1)\rho \geq r.$$

Moreover these contributions depend only on Q_ℓ, $\ell \leq j-1$.

Consider the general term of (3.2.11). The second remark following definition 2.1.1 implies that $[D_t, Q_{j_2}] \in \Sigma^{-(j_2+1)\rho}(j_2+1,\sigma,q)$. Consequently, using again (ii) of proposition 2.1.4, we may write (3.2.11) as a contribution to A_j, plus a remainder belonging to $\mathcal{R}_0^{r_1}(\min(N+1,j-1),\sigma,q)$, depending only on Q_ℓ, $\ell \leq j-2$.

Let us finally consider (3.2.12). If $C = (C_{ij}(U,\omega,\epsilon))_{1\leq i,j\leq 2}$ is an element of $\mathcal{L}_\rho^m(N,\sigma,q)$, $[I',C] = \left[\begin{smallmatrix} 0 & 2c_{12} \\ -2c_{21} & 0 \end{smallmatrix}\right]$ belongs to $\Sigma^{m-2}(N,\sigma,q)$ according to (3.1.3). Consequently, the first term in the sum (3.2.12) is given by the composition of an element in $\Sigma^{-(j_1+1)\rho}(j_1,\sigma,q)$ and of an element in $\Sigma^{-j_2\rho}(j_2,\sigma,q)$. Applying again proposition 2.1.4, we may write this as a contribution to A_j plus a remainder in $\mathcal{R}_0^r(\min(N,j-1),\sigma,q)$, depending only on Q_ℓ, $\ell \leq j-1$. The second term in the argument of the sum (3.2.12) is treated in the same way. This shows that the sum of the first two lines in (3.2.9) contributes to $A+R$ in the right hand side of (3.2.7), since for $j \geq N+1$, A_j is in $\Sigma^{-(N+1)\rho}(N+1,\sigma,q)$, hence in $\mathcal{R}_0^r(N+1,\sigma,q)$ by the inequality $(N+1)\rho \geq r$ and the remark after the statement of definition 2.1.3. Let us show that the last line in (3.2.9) contributes to $B^*\widetilde{L}_\omega + \widetilde{L}_\omega B + R$ in (3.2.7). We have seen above that since Q'_j is in $\mathcal{L}_\rho^{-j\rho}(j,\sigma,q)$ (resp. $Q''_j \in \mathcal{L}_\rho^{-(j+1)\rho}(j,\sigma,q)$), ,$[Q'_j, I'] = \left[\begin{smallmatrix} 0 & e_1 \\ e_2 & 0 \end{smallmatrix}\right]$ with $e_\ell \in \Psi^{-j\rho-2}(j,\sigma,q)$ (resp. $[Q''_j, I'] = \left[\begin{smallmatrix} 0 & e_1 \\ e_2 & 0 \end{smallmatrix}\right]$ with $e_\ell \in \Psi^{-(j+1)\rho-2}(j,\sigma,q)$). We set

$$\begin{aligned}\tilde{B}_j = \sum_{\substack{j_1+j_2=j\\0\leq j_1,j_2\leq N}} I'[Q'^*_{j_1}, I']Q'_{j_2} &+ \sum_{\substack{j_1+j_2=j-1\\0\leq j_1,j_2\leq N}} I'([Q'^*_{j_1}, I']Q''_{j_2} + [Q''^*_{j_1}, I']Q'_{j_2})\\ &+ \sum_{\substack{j_1+j_2=j-2\\0\leq j_1,j_2\leq N}} I'[Q''^*_{j_1}, I']Q''_{j_2}.\end{aligned}$$

Applying proposition 2.1.4, we decompose again $\tilde{B}_j = B_j + R_j$, where B_j belongs to the class $\mathcal{L}_\rho^{-(j+1)\rho}(\min(N,j),\sigma,q)$ (actually, B_j is in $\Sigma^{-(j+1)\rho-2}(\min(N,j),\sigma,q)$) and R_j belongs to $\mathcal{R}_0^{r+2}(\min(j,N),\sigma,q)$ because of (3.2.1). Moreover, B_j depends only on Q'_ℓ, $\ell \leq j$, Q''_ℓ, $\ell \leq j-1$ and by construction, $[\Delta, B_j] \in \Sigma^{-(j+1)\rho}(\min(N,j),\sigma,q)$. For $j \leq N-1$, we get contributions to B and R in (3.2.8), noting that $R_j\widetilde{L}_\omega, \widetilde{L}_\omega R_j$ are in $\mathcal{R}_2^r(N,\sigma,q)$. For $j \geq N$, B_j as well as R_j contribute to the remainder in (3.2.7) since $(N+1)\rho \geq r$. This concludes the proof of (i).

(ii) We write

$$Q^*\widetilde{L}_\omega Q = \frac{1}{2}[Q^*Q\widetilde{L}_\omega + \widetilde{L}_\omega Q^*Q] + \frac{1}{2}[Q^*[\widetilde{L}_\omega, Q] + [Q^*, \widetilde{L}_\omega]Q].$$

By (i), the last term may be written as a contribution to the right hand side of (3.2.8). Let us write the first term in the right hand side under the form $B^*\widetilde{L}_\omega + \widetilde{L}_\omega B + R$. We write Q^*Q as the sum in j of

$$\sum_{\substack{j_1+j_2=j\\0\leq j_1,j_2\leq N}} Q'^*_{j_1}Q'_{j_2} + \sum_{\substack{j_1+j_2=j-1\\0\leq j_1,j_2\leq N}} (Q'^*_{j_1}Q''_{j_2} + Q''^*_{j_1}Q'_{j_2}) + \sum_{\substack{j_1+j_2=j-2\\0\leq j_1,j_2\leq N}} Q''^*_{j_1}Q''_{j_2}.$$

By (3.2.4) and the remark following definition 3.1.2, this may be written $B_j + R_j$ with $B_j \in \mathcal{L}_\rho^{-(j+1)\rho}(\min(N,j),\sigma,q)$ depending only on Q'_ℓ, $\ell \leq j$, Q''_ℓ, $\ell \leq j-1$, $[B_j,\Delta]$ belonging to $\Sigma^{-(j+1)\rho}(\min(N,j),\sigma,q)$, and with $R_j \in \mathcal{R}_0^{r_2}(\min(N,j),\sigma,q)$ with

$$r_2 = \sigma - \sigma_0 - (d+1) + (j+2)\rho - 2\min(j,N) \geq r+2.$$

We obtain contributions to the right hand side of (3.2.8) when $j \leq N-1$, and to the remainder R when $j \geq N$ since $(N+1)\rho \geq r+2$. This concludes the proof. □

Proof of Proposition 3.2.2: We write the left hand side of (3.2.6)

$$\begin{aligned}
\widetilde{L}_\omega + \epsilon V(U,\omega,\epsilon) &+ \epsilon[Q'^*(-\Delta+\mu) + (-\Delta+\mu)Q'] \\
&+ \epsilon[Q''^*\widetilde{L}_\omega + \widetilde{L}_\omega Q''] \\
&+ \epsilon[Q'^* I'\omega D_t + \omega I' D_t Q'] \\
&+ \epsilon^2 Q^*\widetilde{L}_\omega Q + \epsilon^2[Q^*V + VQ] + \epsilon^3 Q^*VQ.
\end{aligned} \tag{3.2.13}$$

The term V in (3.2.13) contributes to the V_0 component of V^N in the right hand side of (3.2.6). The first two brackets in (3.2.13) give rise to the last two ones in (3.2.6). To study the contribution of $Q^*\widetilde{L}_\omega Q$, we use (3.2.8). The B_j component of B in the right hand side of (3.2.8) contributes to the S_j component of S^N in (3.2.6). Let us study the third bracket in (3.2.13). By (3.2.4) and definition 3.1.2, we may write $Q'_{j-1} = \left[\begin{smallmatrix} a & b \\ b^* & c\end{smallmatrix}\right]$ with $a, c \in \Psi^{-j\rho}(j-1,\sigma,q)$, $b \in \Psi^{-(j-1)\rho-2}(j-1,\sigma,q)$, $a^* = -a$, $c^* = -c$. This implies that

$$Q'^*_{j-1} I' D_t + I' D_t Q'_{j-1} = \begin{bmatrix} [D_t, a] & [D_t, b] \\ -[D_t, b^*] & -[D_t, c]\end{bmatrix}$$

is a self-adjoint operator belonging to $\Sigma^{-j\rho}(j,\sigma,q)$, $1 \leq j \leq N$ using the second remark after definition 2.1.1. We thus get a contribution to V_j in (3.2.6).

Finally, let us check that the last two terms in (3.2.13) may be written as contributions to V^N and to R in the right hand side of (3.2.6). Actually, we may write $Q^*V + VQ + \epsilon Q^*VQ$ as the sum in j of

$$\begin{aligned}
Q'^*_{j-1}V + VQ'_{j-1} &+ Q''^*_{j-2}V + VQ''_{j-2} \\
&+\epsilon \sum_{j_1+j_2=j-2} Q'^*_{j_1} V Q'_{j_2} + \epsilon \sum_{j_1+j_2=j-3} (Q''^*_{j_1} V Q'_{j_2} + Q'^*_{j_1} V Q''_{j_2}) \\
&+ \epsilon \sum_{j_1+j_2=j-4} Q''^*_{j_1} V Q''_{j_2}.
\end{aligned} \tag{3.2.14}$$

Using that $Q'_j \in \Sigma^{-(j+1)\rho}(j,\sigma,q)$, $Q''_j \in \Sigma^{-(j+2)\rho}(j,\sigma,q)$, $V \in \Sigma^0(0,\sigma,q)$, we write (3.2.14) as $V_j + R_j$ where V_j depends only on Q'_ℓ, $\ell \leq j-1$, Q''_ℓ, $\ell \leq j-2$ and is in $\Sigma^{-j\rho}(\min(N,j-1),\sigma,q)$ and $R_j \in \mathcal{R}_0^r(N,\sigma,q)$. This concludes the proof. □

Proof of Proposition 3.2.1: Let us construct recursively Q'_j, $0 \leq j \leq N$, Q''_j, $0 \leq j \leq N-1$ so that the right hand side of (3.2.6) may be written as the right hand side of (3.2.3). Assume that

$Q_0, \dots, Q_{j-1}$ have been already determined in such a way that the right hand side of (3.2.6) may be written

$$
\begin{aligned}
&\widetilde{L}_\omega + \epsilon \sum_{j'=0}^{j-1} V_{\mathrm{D},j'} + \epsilon \sum_{j'=j}^{N-1} [S_{j'}^* \widetilde{L}_\omega + \widetilde{L}_\omega S_{j'}] \\
&\quad + \epsilon \sum_{j'=j}^{N} [Q_{j'}'^*(-\Delta + \mu) + (-\Delta + \mu) Q_{j'}'] \\
&\quad + \epsilon \sum_{j'=j}^{N-1} [Q_{j'}''^* \widetilde{L}_\omega + \widetilde{L}_\omega Q_{j'}''] \\
&\quad + \epsilon \sum_{j'=j}^{N} V_{j'} + \epsilon R.
\end{aligned}
\tag{3.2.15}
$$

Write $V_j = \left[\begin{smallmatrix} a & b \\ b^* & c \end{smallmatrix}\right]$ with $a, b, c \in \Psi^{-j\rho}(j, \sigma, q)$, $a^* = a$, $c^* = c$ and define

$$
V_{\mathrm{D},j} = \sum_{\alpha \in \mathcal{A}} \widetilde{\Pi}_\alpha \left[\begin{smallmatrix} a & 0 \\ 0 & c \end{smallmatrix}\right] \widetilde{\Pi}_\alpha, \ V_{\mathrm{ND},j} = V_j - V_{\mathrm{D},j}.
$$

Then $V_{\mathrm{D},j} \in \Sigma_{\mathrm{D}}^{-j\rho}(j, \sigma, q)$, $(V_{\mathrm{D},j})^* = V_{\mathrm{D},j}$ and $V_{\mathrm{ND},j}$ is in $\Sigma_{\mathrm{ND}}^{-j\rho}(j, \sigma, q)$, $(V_{\mathrm{ND},j})^* = V_{\mathrm{ND},j}$. Moreover $V_{\mathrm{ND},j}$ depends only on Q_ℓ, $\ell \leq j-1$. We apply proposition 3.1.3 to find $Q_j' \in \mathcal{L}_\rho'^{-j\rho}(j, \sigma, q)$ and $R_j \in \mathcal{R}_0^{r(\sigma,j)+j\rho}(j, \sigma, q)$ such that $Q_j'^*(-\Delta + \mu) + (-\Delta + \mu) Q_j' = V_{\mathrm{ND},j} + R_j$ and $[\Delta, Q_j']$ is in $\Sigma^{-j\rho}(j, \sigma, q)$. The assumption (3.2.1) on σ shows that R_j contributes to R_1 in (3.2.3). Moreover condition (3.2.4) is satisfied by Q_j', so that we have eliminated the jth component in the fourth and sixth terms of (3.2.15). To eliminate the jth component of the third and fifth terms, we set $Q_j'' = -S_j$, $j \leq N-1$, $Q_N'' = 0$. Then condition (3.2.4) is satisfied by Q_j'', and the definition is consistent since S_j depends only on Q_ℓ', $\ell \leq j$, Q_ℓ'', $\ell \leq j-1$. This concludes the proof. □

4 Iterative scheme

This section will be devoted to the proof of theorem 1.1.1. We shall construct a solution to equation (2.3.15) – which is equivalent to equation (1.1.3) – writing this equation under an equivalent form involving the right hand side of (3.2.3). The first subsection will be devoted to the study of the restriction of the operator $\widetilde{L}_\omega + \epsilon V_{\mathrm{D}}(U, \omega, \epsilon)$ to the range of one of the projectors $\widetilde{\Pi}_\alpha$. We shall show that, for (ω, ϵ) outside a subset of small measure, this restriction is invertible. As usual in these problems, the inverse we construct loses derivatives. This will not cause much trouble, since proposition 3.2.1 allows us to write the equation essentially under the form $(\widetilde{L}_\omega + \epsilon V_{\mathrm{D}}(U, \omega, \epsilon)) W = \epsilon R_1(U, \omega, \epsilon) W$ for a new unknown W. Since R_1 is smoothing, it gains enough derivatives to compensate the losses coming from $(\widetilde{L}_\omega + \epsilon V_{\mathrm{D}})^{-1}$. Because of that, we may construct the solution using a standard iterative scheme.

4.1 Lower bounds for eigenvalues

Let $\gamma_0 \in]0,1], \sigma \in \mathbb{R}, N \in \mathbb{N}, \zeta \in \mathbb{R}_+$ such that $\sigma \geq \sigma_0 + \frac{\zeta}{\rho} + 2(N+1) + d + 1$. We denote by $\mathcal{E}^\sigma(\zeta)$ the space of functions

$$\begin{aligned}(t,x,\omega,\epsilon) &\longrightarrow U(t,x,\omega,\epsilon)\\ \mathbb{S}^1 \times \mathbb{T}^d \times [1,2] \times [0,\gamma_0] &\to \mathbb{C}^2\end{aligned} \tag{4.1.1}$$

which are continuous functions of ω with values in $\mathcal{H}^\sigma(\mathbb{S}^1 \times \mathbb{T}^d; \mathbb{C}^2)$ and C^1 functions of ω with values in $\mathcal{H}^{\sigma-\zeta-2}(\mathbb{S}^1 \times \mathbb{T}^d; \mathbb{C}^2)$, uniformly in $\epsilon \in [0,\gamma_0]$. We set

$$\|U\|_{\mathcal{E}^\sigma(\zeta)} = \sup_{(\omega,\epsilon)\in[1,2]\times[0,\gamma_0]} \|U(\cdot,\omega,\epsilon)\|_{\mathcal{H}^\sigma} + \sup_{(\omega,\epsilon)\in[1,2]\times[0,\gamma_0]} \|\partial_\omega U(\cdot,\omega,\epsilon)\|_{\mathcal{H}^{\sigma-\zeta-2}}. \tag{4.1.2}$$

If $\widetilde{\Pi}_\alpha$ is the projector of $\mathcal{H}^0$ given by (1.2.5), we set $F_\alpha = \operatorname{Range}(\widetilde{\Pi}_\alpha)$, $D_\alpha = \dim F_\alpha$. By (1.2.4) and (1.2.6), $D_\alpha \leq C_1 \langle n(\alpha)\rangle^{\beta d+2}$ for some $C_1 > 0$. We define for $U \in \mathcal{E}^\sigma(\zeta)$, $\omega \in [1,2]$, $\epsilon \in [0,\gamma_0]$

$$A_\alpha(\omega; U, \epsilon) = \widetilde{\Pi}_\alpha(\widetilde{L}_\omega + \epsilon V_{\mathrm{D}}(U,\omega,\epsilon))\widetilde{\Pi}_\alpha. \tag{4.1.3}$$

This is a self-adjoint operator on F_α, with C^1 dependence in ω, since it follows from the expression (3.2.2) of V_{D}, condition (2.1.1) in the definition of $\Psi^m(N,\sigma,q)$, the fact that $\partial_\omega U \in \mathcal{H}^{\sigma-\zeta-2}$, and the assumption made on σ, that $\omega \to \widetilde{\Pi}_\alpha V_{\mathrm{D}}(U(t,x,\omega,\epsilon),\omega,\epsilon)\widetilde{\Pi}_\alpha$ is C^1. The main result of this subsection is the following:

Proposition 4.1.1 *For any $\mu \in \mathbb{R} - \mathbb{Z}_-$, any $q > 0$, there are $\gamma_0 \in]0,1]$, $C_0 > 0$, $\mathcal{A}_0 \subset \mathcal{A}$ a finite subset, and for any $U \in \mathcal{E}^\sigma(\zeta)$ with $\|U\|_{\mathcal{E}^\sigma(\zeta)} < q$, any $\epsilon \in [0,\gamma_0]$, any $\alpha \in \mathcal{A}$, the eigenvalues of A_α form a finite family of C^1 real valued functions of ω, depending on (U,ϵ),*

$$\omega \to \lambda_\ell^\alpha(\omega; U, \epsilon),\ 1 \leq \ell \leq D_\alpha \tag{4.1.4}$$

satisfying the following properties:

(i) For any $\alpha \in \mathcal{A}$, any $U, U' \in \mathcal{H}^\sigma$ with $\|U\|_{\mathcal{H}^\sigma} < q$, $\|U'\|_{\mathcal{H}^\sigma} < q$, any $\ell \in \{1,\dots,D_\alpha\}$, any $\epsilon \in [0,\gamma_0]$, any $\omega \in [1,2]$, there is $\ell' \in \{1,\dots,D_\alpha\}$ such that

$$|\lambda_\ell^\alpha(\omega; U,\epsilon) - \lambda_{\ell'}^\alpha(\omega; U',\epsilon)| \leq C_0 \epsilon \|U - U'\|_{\mathcal{H}^\sigma}. \tag{4.1.5}$$

(ii) For any $a \in \mathcal{A} - \mathcal{A}_0$, any $U \in \mathcal{E}^\sigma(\zeta)$ with $\|U\|_{\mathcal{E}^\sigma(\zeta)} < q$, any $\epsilon \in [0,\gamma_0]$, any $\ell \in \{1,\dots,D_\alpha\}$, either

$$C_0^{-1}\langle n(\alpha)\rangle^2 \leq \frac{\partial \lambda_\ell^\alpha}{\partial \omega}(\omega; U,\epsilon) \leq C_0 \langle n(\alpha)\rangle^2 \textit{ for any } \omega \textit{ in } [1,2], \tag{4.1.6}$$

or

$$-C_0\langle n(\alpha)\rangle^2 \leq \frac{\partial \lambda_\ell^\alpha}{\partial \omega}(\omega; U,\epsilon) \leq -C_0^{-1} \langle n(\alpha)\rangle^2 \textit{ for any } \omega \textit{ in } [1,2]. \tag{4.1.7}$$

(iii) Denote for $\delta \in]0,1]$, $\epsilon \in [0,\gamma_0]$, $\alpha \in \mathcal{A}$, $U \in \mathcal{E}^\sigma(\zeta)$ with $\|U\|_{\mathcal{E}^\sigma(\zeta)} < q$,

$$I(\alpha, U, \epsilon, \delta) = \{\omega \in [1,2]; \forall \ell \in \{1,\dots,D_\alpha\}, |\lambda_\ell^\alpha(\omega; U,\epsilon)| \geq \delta \langle n(\alpha)\rangle^{-\zeta}\}. \tag{4.1.8}$$

Then there is a constant E_0, depending only on the dimension, such that for any $\omega \in I(\alpha, U, \epsilon, \delta)$, $A_\alpha(\omega; U, \epsilon)$ is invertible and

$$\begin{aligned} \|A_\alpha(\omega; U, \epsilon)^{-1}\|_{\mathcal{L}(\mathcal{H}^0)} &\leq E_0 \delta^{-1} \langle n(\alpha)\rangle^{\zeta} \\ \|\partial_\omega A_\alpha(\omega; U, \epsilon)^{-1}\|_{\mathcal{L}(\mathcal{H}^0)} &\leq E_0 \delta^{-2} \langle n(\alpha)\rangle^{2\zeta+2}. \end{aligned} \tag{4.1.9}$$

Proof: The proof of such a result is quite classical, and may be found in the references given in the introduction. For the sake of completeness, we give it in detail.

(i) By construction, A_α is a self-adjoint operator, acting on a space of finite dimension D_α. Moreover, A_α is a C^1 function of ω if $U \in \mathcal{E}^\sigma(\zeta)$. By a theorem of Rellich (see for instance theorem 6.8 in the book of Kato [20]), we know that we may index eigenvalues of that matrix so that they are C^1 functions of ω, $\lambda_\ell^\alpha(\omega; U, \epsilon)$, $1 \leq \ell \leq D_\alpha$. Moreover, if B and B' are two self-adjoint matrices of the same dimension, for any eigenvalue $\lambda_\ell(B)$ of B, there is an eigenvalue $\lambda_{\ell'}(B')$ of B' such that $|\lambda_\ell(B) - \lambda_{\ell'}(B')| \leq \|B - B'\|$. Combining this with the fact that $U \to A_\alpha(\omega; U, \epsilon)$ is lipschitz with values in $\mathcal{L}(\mathcal{H}^0)$, with lipschitz constant $C\epsilon$, we get (4.1.5).

(ii) Set

$$\Lambda^0_\pm(\alpha) = \{\pm j\omega + |n|^2 + \mu; j \in \mathbb{N}, n \in \Omega_\alpha, K_0^{-1}\langle n(\alpha)\rangle^2 \leq j \leq K_0 \langle n(\alpha)\rangle^2\}$$

so that the spectrum of $\widetilde{\Pi}_\alpha \widetilde{L}_\omega \widetilde{\Pi}_\alpha$ is $\Lambda^0_+(\alpha) \cup \Lambda^0_-(\alpha)$. The difference between an eigenvalue in $\Lambda^0_+(\alpha)$, parametrized by (j, n), and an eigenvalue in $\Lambda^0_-(\alpha)$, parametrized by (j', n') $(j > 0, j' < 0)$ is bounded from below by

$$\omega(j - j') + |n|^2 - |n'|^2 \geq 2K_0^{-1}\langle n(\alpha)\rangle^2 - \theta - C\langle n(\alpha)\rangle^\beta$$

by the first estimate (1.2.2), for some $C > 0, \beta \in]0, \frac{1}{10}[$. If we take the subset $\mathcal{A}_0$ large enough, we get that when $\alpha \in \mathcal{A} - \mathcal{A}_0$, the difference between such two eigenvalues is bounded from below by $K_0^{-1}\langle n(\alpha)\rangle^2$. Consequently, if $0 \leq \epsilon < \gamma_0$ small enough, the spectrum of A_α may be split in two subsets $\Lambda_+(\alpha) \cup \Lambda_-(\alpha)$ whose distance is bounded from below by $\frac{1}{2}K_0^{-1}\langle n(\alpha)\rangle^2$. Let Γ be a contour in the complex plane turning once around $\Lambda^0_+(\alpha)$, of length $O(\langle n(\alpha)\rangle^2)$, such that the distance between Γ and the spectrum of $\widetilde{L}^\alpha_\omega = \widetilde{\Pi}_\alpha \widetilde{L}_\omega \widetilde{\Pi}_\alpha$ is bounded from below by $c\langle n(\alpha)\rangle^2$, and such that $\Lambda^0_-(\alpha)$ is outside Γ. If γ_0 is small enough, this contour satisfies the same conditions with $\Lambda^0_\pm(\alpha)$ replaced by $\Lambda_\pm(\alpha)$ and $\widetilde{L}^\alpha_\omega$ replaced by A_α. The spectral projector $\widetilde{\Pi}^+_\alpha(\omega)$ (resp. $\widetilde{\Pi}^{+,0}_\alpha$) associated to the eigenvalues $\Lambda_+(\alpha)$ (resp. $\Lambda^0_+(\alpha)$) of A_α (resp. $\widetilde{L}^\alpha_\omega$) is given by

$$\widetilde{\Pi}^+_\alpha(\omega) = \frac{1}{2i\pi}\int_\Gamma (\zeta \mathrm{Id} - A_\alpha)^{-1}\, d\zeta,\ \widetilde{\Pi}^{+,0}_\alpha = \frac{1}{2i\pi}\int_\Gamma (\zeta \mathrm{Id} - \widetilde{L}^\alpha_\omega)^{-1}\, d\zeta. \tag{4.1.10}$$

Note that the second projector is just the orthogonal projector on

$$\mathrm{Vect}\,\{e^{i(jt+n\cdot x)}; n \in \Omega_\alpha, K_0^{-1}\langle n(\alpha)\rangle^2 \leq j \leq K_0\langle n(\alpha)\rangle^2\},$$

so is independent of ω. Write

$$\widetilde{\Pi}^+_\alpha(\omega) - \widetilde{\Pi}^{+,0}_\alpha = \frac{1}{2i\pi}\int_\Gamma (\zeta \mathrm{Id} - A_\alpha)^{-1}(A_\alpha - \widetilde{L}^\alpha_\omega)(\zeta \mathrm{Id} - \widetilde{L}^\alpha_\omega)^{-1}\, d\zeta. \tag{4.1.11}$$

Using (4.1.3) and the definition of $\widetilde{L}^\alpha_\omega$

$$\begin{aligned}&\|A_\alpha - \widetilde{L}^\alpha_\omega\|_{\mathcal{L}(F_\alpha)} + \|\partial_\omega(A_\alpha - \widetilde{L}^\alpha_\omega)\|_{\mathcal{L}(F_\alpha)} \le C\epsilon\\ &\|\partial_\omega A_\alpha\|_{\mathcal{L}(F_\alpha)} + \|\partial_\omega \widetilde{L}^\alpha_\omega\|_{\mathcal{L}(F_\alpha)} \le C\langle n(\alpha)\rangle^2.\end{aligned}$$

Consequently (4.1.11) implies

$$\begin{aligned}&\|\widetilde{\Pi}^+_\alpha(\omega) - \widetilde{\Pi}^{+,0}_\alpha\|_{\mathcal{L}(F_\alpha)} \le C\epsilon\langle n(\alpha)\rangle^{-2}\\ &\|\partial_\omega \widetilde{\Pi}^+_\alpha(\omega)\|_{\mathcal{L}(F_\alpha)} = \|\partial_\omega(\widetilde{\Pi}^+_\alpha(\omega) - \widetilde{\Pi}^{+,0}_\alpha)\|_{\mathcal{L}(F_\alpha)} \le C\epsilon\langle n(\alpha)\rangle^{-2}.\end{aligned}$$

Writing

$$\begin{aligned}\widetilde{\Pi}^+_\alpha(\omega)A_\alpha\widetilde{\Pi}^+_\alpha(\omega) = (\widetilde{\Pi}^+_\alpha(\omega) - \widetilde{\Pi}^{+,0}_\alpha)A_\alpha\widetilde{\Pi}^+_\alpha(\omega) + \widetilde{\Pi}^{+,0}_\alpha(A_\alpha - \widetilde{L}^\alpha_\omega)\widetilde{\Pi}^+_\alpha(\omega)\\ +\widetilde{\Pi}^{+,0}_\alpha\widetilde{L}^\alpha_\omega(\widetilde{\Pi}^+_\alpha(\omega) - \widetilde{\Pi}^{+,0}_\alpha) + \widetilde{\Pi}^{+,0}_\alpha\widetilde{L}^\alpha_\omega\widetilde{\Pi}^{+,0}_\alpha\end{aligned}$$

we obtain that

$$\|\partial_\omega[\widetilde{\Pi}^+_\alpha(\omega)A_\alpha\widetilde{\Pi}^+_\alpha(\omega) - \widetilde{\Pi}^{+,0}_\alpha\widetilde{L}^\alpha_\omega\widetilde{\Pi}^{+,0}_\alpha]\|_{\mathcal{L}(F_\alpha)} \le C\epsilon. \tag{4.1.12}$$

Let I be an interval contained in $[1,2]$ over which one of the eigenvalue $\lambda^\alpha_\ell(\omega; U, \epsilon)$ of the matrix $\widetilde{\Pi}^+_\alpha(\omega)A_\alpha(\omega; U, \epsilon)\widetilde{\Pi}^+_\alpha(\omega)$ has constant multiplicity m, denote by $P(\omega)$ the associated spectral projector. Then $P(\omega)$ is C^1 in $\omega \in I$ and satisfies $P(\omega)^2 = P(\omega)$, whence $P(\omega)P'(\omega)P(\omega) = 0$. We get therefore for

$$\lambda^\alpha_\ell(\omega; U, \epsilon) = \frac{1}{m}\mathrm{tr}[P(\omega)\widetilde{\Pi}^+_\alpha(\omega)A_\alpha(\omega; U, \epsilon)\widetilde{\Pi}^+_\alpha(\omega)P(\omega)]$$

the equality

$$\partial_\omega\lambda^\alpha_\ell(\omega; U, \epsilon) = \frac{1}{m}\mathrm{tr}[P(\omega)\partial_\omega(\widetilde{\Pi}^+_\alpha(\omega)A_\alpha(\omega; U, \epsilon)\widetilde{\Pi}^+_\alpha(\omega))P(\omega)].$$

By (4.1.12), we obtain

$$\partial_\omega\lambda^\alpha_\ell(\omega; U, \epsilon) = \frac{1}{m}\mathrm{tr}[P(\omega)\partial_\omega(\widetilde{\Pi}^{+,0}_\alpha\widetilde{L}^\alpha_\omega\widetilde{\Pi}^{+,0}_\alpha)P(\omega)] + O(\epsilon). \tag{4.1.13}$$

Since $\widetilde{\Pi}^{+,0}_\alpha\widetilde{L}^\alpha_\omega\widetilde{\Pi}^{+,0}_\alpha$ is by definition of $\widetilde{L}^\alpha_\omega$ a diagonal matrix with entries $j\omega + |n|^2 + \mu$, $n \in \Omega_\alpha$, $K_0^{-1}\langle n(\alpha)\rangle^2 \le j \le K_0\langle n(\alpha)\rangle^2$, we see that (4.1.13) stays between $K_0^{-1}\langle n(\alpha)\rangle^2 - C\epsilon$ and $K_0\langle n(\alpha)\rangle^2 + C\epsilon$. This implies (4.1.6) if $\epsilon \in [0, \gamma_0]$ with γ_0 small enough. The case of eigenvalues corresponding to $\Lambda_-(\alpha)$ is treated in a similar way, and gives (4.1.7).

(iii) The first estimate in (4.1.9) follows from the fact that the eigenvalues $\lambda^\alpha_\ell(\omega; U, \epsilon)$ of A_α satisfy the lower bound given by the definition of (4.1.8). The second estimate is a consequence of the first one and of the fact that $\|\partial_\omega A_\alpha(\omega; U, \epsilon)\|_{\mathcal{L}(\mathcal{H}^0)} \le C\langle n(\alpha)\rangle^2$ by definition of A_α. This concludes the proof. $\square$

4.2 Iterative scheme

This subsection will be devoted to the proof of theorem 1.1.1, constructing the solution as the limit of an iterative scheme. We fix indices $s, \sigma, N, \zeta, r, \delta$ satisfying the following inequalities

$$
\begin{aligned}
&\sigma \geq \sigma_0 + 2(N+1) + d + 1 + r/\rho,\ r = \zeta \\
&(N+1)\rho \geq r + 2,\ s \geq \sigma + \zeta + 2,\ \delta \in]0, \delta_0],
\end{aligned} \tag{4.2.1}
$$

where $\delta_0 > 0$ will be chosen small enough. We also assume that the parameter μ is in $\mathbb{R} - \mathbb{Z}_-$. We shall solve equation (2.3.15) when its force term f is given in $\mathcal{H}^{s+\zeta}(\mathbb{S}^1 \times \mathbb{T}^d; \mathbb{C}^2)$. To achieve this goal, the main task will be to construct a sequence $(G_k, \mathcal{O}_k, \psi_k, U_k, W_k)$, $k \geq 0$, where $G_k, \mathcal{O}_k$ will be subsets of $[1,2] \times [0, \delta^2]$, ψ_k will be a real valued function defined on $[1,2] \times [0, \delta^2]$, U_k, W_k will be functions of $(t, x, \omega, \epsilon) \in \mathbb{S}^1 \times \mathbb{T}^d \times [1,2] \times [0, \delta^2]$ with values in $\mathbb{C}^2$. At order $k = 0$, we define

$$
\begin{aligned}
&U_0 = W_0 = 0 \\
&\mathcal{O}_0 = \{(\omega, \epsilon) \in [1,2] \times [0, \gamma_0]; \exists \alpha \in \mathcal{A}_0, \exists \ell \in \{1, \dots, D_\alpha\} \text{ with } |\lambda_\alpha^\ell(\omega; 0, \epsilon)| < 2\delta\}
\end{aligned} \tag{4.2.2}
$$

using the notations of proposition 4.1.1. For any $\epsilon \in [0, \gamma_0]$ we denote by $\mathcal{O}_{0,\epsilon}$ the ϵ-section of $\mathcal{O}_0$ and set

$$
G_0 = \Big\{(\omega, \epsilon) \in [1,2] \times [0, \gamma_0]; d(\omega, \mathbb{R} - \mathcal{O}_{0,\epsilon}) \geq \frac{\delta}{8C_0'}\Big\}
$$

where $C_0' > 0$ is a constant such that for any $\alpha \in \mathcal{A}_0$, any $\ell \in \{1, \dots, D_\alpha\}$, any $(\omega, \epsilon) \in [1,2] \times [0, \gamma_0]$, $|\partial_\omega \lambda_\ell^\alpha(\omega; 0, \epsilon)| \leq C_0'$. Then $\mathcal{O}_0$ is an open subset of $[1,2] \times [0, \gamma_0]$ and for any $\epsilon \in [0, \gamma_0]$, $G_{0,\epsilon}$ is a closed subset of $[1,2]$, contained in the open subset $\mathcal{O}_{0,\epsilon}$. By Urysohn's lemma, we may for each fixed ϵ construct a C^1 function $\omega \to \psi_0(\omega, \epsilon)$, compactly supported in $\mathcal{O}_{0,\epsilon}$, equal to one on $G_{0,\epsilon}$, such that for any ω, ϵ, $0 \leq \psi_0(\omega, \epsilon) \leq 1$, $|\partial_\omega \psi_0(\omega, \epsilon)| \leq C_1 \delta^{-1}$ for some uniform constant C_1 depending only on C_0'.

We denote by

$$
\widetilde{S}_k = \sum_{\alpha \in \mathcal{A}; \langle n(\alpha) \rangle < 2^k} \widetilde{\Pi}_\alpha,\ k \geq 1. \tag{4.2.3}
$$

Proposition 4.2.1 *There are $\delta_0 \in]0, \sqrt{\gamma_0}]$, positive constants C_1, B_1, B_2 and for any $k \geq 1$, any $\delta \in]0, \delta_0]$, a 5-uple $(G_k, \mathcal{O}_k, \psi_k, U_k, W_k)$ satisfying the following conditions for any $\delta \in]0, \delta_0]$:*

$$
\begin{aligned}
\mathcal{O}_k = \{(\omega, \epsilon) \in [1,2] \times [0, \delta^2]; \exists \alpha \in \mathcal{A} - \mathcal{A}_0 \text{ with } 2^{k-1} \leq \langle n(\alpha) \rangle < 2^k, \\
\exists \ell \in \{1, \dots, D_\alpha\} \text{ with } |\lambda_\alpha^\ell(\omega; U_{k-1}, \epsilon)| < 2\delta 2^{-k\zeta}\}, \\
G_k = \{(\omega, \epsilon) \in [1,2] \times [0, \delta^2]; d(\omega, \mathbb{R} - \mathcal{O}_{k,\epsilon}) \geq \frac{\delta}{8C_0} 2^{-k(\zeta+2)}\},
\end{aligned} \tag{4.2.4}
$$

where C_0 is the constant in (4.1.6), (4.1.7);

$$
\begin{aligned}
\psi_k : [1,2] \times [0, \delta^2] \to [0,1] \text{ is supported in } \mathcal{O}_k, \text{ equal to 1 on } G_k, \\
C^1 \text{ in } \omega \text{ and for all } (\omega, \epsilon), |\partial_\omega \psi_k(\omega, \epsilon)| \leq \frac{C_1}{\delta} 2^{k(\zeta+2)};
\end{aligned} \tag{4.2.5}
$$

The function $(t,x,\omega,\epsilon)\to W_k(t,x,\omega,\epsilon)$ is for any $\epsilon\in[0,\delta^2]$ a continuous function of ω with values in $\mathcal{H}^s(\mathbb{S}^1\times\mathbb{T}^d;\mathbb{C}^2)$, which is a C^1 function of ω with values in $\mathcal{H}^{s-\zeta-2}(\mathbb{S}^1\times\mathbb{T}^d;\mathbb{C}^2)$ satisfying

$$\|W_k(\cdot,\omega,\epsilon)\|_{\mathcal{H}^s}+\delta\|\partial_\omega W_k(\cdot,\omega,\epsilon)\|_{\mathcal{H}^{s-\zeta-2}}\leq B_1\frac{\epsilon}{\delta} \tag{4.2.6}$$

uniformly in $\epsilon\in[0,\delta^2],\omega\in[1,2],\delta\in]0,\delta_0]$. Moreover, for any $(\omega,\epsilon)\in[1,2]\times[0,\delta^2]-\bigcup_{k'=0}^{k}\mathcal{O}_{k'}$, W_k solves the equation

$$\begin{aligned}(\widetilde{L}_\omega+\epsilon V_{\mathrm{D}}(U_{k-1},\omega,\epsilon))W_k=&\epsilon\widetilde{S}_k(\mathrm{Id}+\epsilon Q(U_{k-1},\omega,\epsilon))^*R(U_{k-1},\omega,\epsilon)U_{k-1}\\&+\epsilon\widetilde{S}_k[R_1(U_{k-1},\omega,\epsilon)W_{k-1}]\\&+\epsilon\widetilde{S}_k(\mathrm{Id}+\epsilon Q(U_{k-1},\omega,\epsilon))^*f\end{aligned} \tag{4.2.7}$$

where R is defined by the right hand side of (2.3.15) and Q,V_{D},R_1 are defined in (3.2.2), (3.2.3);

The function U_k is defined from W_k by

$$U_k(t,x,\omega,\epsilon)=(\mathrm{Id}+\epsilon Q(U_{k-1},\omega,\epsilon))W_k \tag{4.2.8}$$

and it satisfies

$$\begin{aligned}&\|U_k-U_{k-1}\|_{\mathcal{H}^\sigma}\leq 2B_2\frac{\epsilon}{\delta}2^{-k\zeta}\\&\|U_k(\cdot,\omega,\epsilon)\|_{\mathcal{H}^s}+\delta\|\partial_\omega U_k(\cdot,\omega,\epsilon)\|_{\mathcal{H}^{s-\zeta-2}}\leq B_2\frac{\epsilon}{\delta}\end{aligned} \tag{4.2.9}$$

uniformly for $\omega\in[1,2],\epsilon\in[0,\delta^2],\delta\in]0,\delta_0]$. Moreover

$$\|W_k-W_{k-1}\|_{\mathcal{H}^\sigma}\leq B_2\frac{\epsilon}{\delta}2^{-k\zeta}. \tag{4.2.10}$$

Remark: Note that since we assume $\epsilon\leq\delta^2$, the second estimate (4.2.9) implies, with the notation introduced in (4.1.2), the uniform bound

$$\|U_k\|_{\mathcal{E}^\sigma(\zeta)}<q \tag{4.2.11}$$

for some q.

Let us write the equation for U_k following from (4.2.8) and (4.2.7). Because of the uniform estimate (4.2.11) for U_{k-1}, if $0\leq\epsilon\leq\delta^2\leq\delta_0^2$ with δ_0 small enough $(\mathrm{Id}+\epsilon Q(U_{k-1},\omega,\epsilon))^*$ is invertible for any $(\omega,\epsilon)\in[1,2]\times[0,\delta^2]$. If we write

$$(\widetilde{L}_\omega+\epsilon V(U_{k-1},\omega,\epsilon))U_k=(\widetilde{L}_\omega+\epsilon V(U_{k-1},\omega,\epsilon))(\mathrm{Id}+\epsilon Q(U_{k-1},\omega,\epsilon))W_k$$

and if we use (3.2.3) multiplied on the left by $(\mathrm{Id}+\epsilon Q(U_{k-1},\omega,\epsilon)^*)^{-1}$ and (4.2.7), we get

$$\begin{aligned}(\widetilde{L}_\omega+\epsilon V(U_{k-1},\omega,\epsilon))U_k=\epsilon(\mathrm{Id}+\epsilon Q(U_{k-1},\omega,\epsilon)^*)^{-1}[&\widetilde{S}_k(\mathrm{Id}+\epsilon Q(U_{k-1},\omega,\epsilon)^*)R(U_{k-1},\omega,\epsilon)U_{k-1}\\&+\widetilde{S}_kR_1(U_{k-1},\omega,\epsilon)W_{k-1}\\&+\widetilde{S}_k(\mathrm{Id}+\epsilon Q(U_{k-1},\omega,\epsilon)^*)f\\&-R_1(U_{k-1},\omega,\epsilon)W_k]\end{aligned} \tag{4.2.12}$$

for any $(\omega, \epsilon) \in [1,2] \times [0, \delta^2] - \bigcup_{k'=0}^{k} \mathcal{O}_{k'}$, $\delta \in [0, \delta_0]$.

Proof of Proposition 4.2.1: We assume that $(G_k, \mathcal{O}_k, \psi_k, U_k, W_k)$ have been constructed satisfying (4.2.4) to (4.2.9), and shall construct these data at rank $k+1$, if δ_0 is small enough and the constants C_1, B_1, B_2 are large enough.

The sets $\mathcal{O}_{k+1}, G_{k+1}$ are defined by (4.2.4) at rank $k+1$ as soon as U_k is given. Then for fixed ϵ, $G_{k+1,\epsilon}$ is a compact subset of the open set $\mathcal{O}_{k+1,\epsilon}$, whose distance to the complement of $\mathcal{O}_{k+1,\epsilon}$ is bounded from below by $\frac{\delta}{8C_0} 2^{-(k+1)(\zeta+2)}$. We may construct by Urysohn's lemma a function ψ_{k+1} satisfying (4.2.5) at rank $k+1$. Let us construct W_{k+1} for $(\omega, \epsilon) \in [1,2] \times [0, \delta^2] - \bigcup_{k'=0}^{k+1} G_{k'}$. Since $V_{\mathrm{D}}(U_k, \omega, \epsilon)$ is by construction a block-diagonal operator, we may write equation (4.2.7) at rank $k+1$ as the following system of equations:

$$
\begin{aligned}
(\widetilde{L}_\omega + \epsilon V_{\mathrm{D}}(U_k, \omega, \epsilon)) \widetilde{\Pi}_\alpha W_{k+1} = \epsilon \widetilde{\Pi}_\alpha \widetilde{S}_{k+1} (\mathrm{Id} + \epsilon Q(U_k, \omega, \epsilon)^*) R(U_k, \omega, \epsilon) U_k \\
+ \epsilon \widetilde{\Pi}_\alpha \widetilde{S}_{k+1} R_1(U_k, \omega, \epsilon) W_k \\
+ \epsilon \widetilde{\Pi}_\alpha \widetilde{S}_{k+1} (\mathrm{Id} + \epsilon Q(U_k, \omega, \epsilon)^*) f
\end{aligned}
\tag{4.2.13}
$$

for any $\alpha \in \mathcal{A}$. If $\langle n(\alpha) \rangle \geq 2^{k+1}$, the right hand side of (4.2.13) vanishes by definition of $\widetilde{S}_{k+1}$, so that we may set in this case $\widetilde{\Pi}_\alpha W_{k+1} = 0$ by definition. Let us solve (4.2.13) for those α satisfying $\langle n(\alpha) \rangle < 2^{k+1}$. We shall apply proposition 4.1.1, using the following lemma:

Lemma 4.2.2 *There is $\delta_0 \in]0,1]$, depending only on the constants B_1, B_2, such that for any $k \geq 0$, any $k' \in \{1, \ldots, k+1\}$, any $\delta \in [0, \delta_0]$, any $\epsilon \in [0, \delta^2]$, any $\alpha \in \mathcal{A} - \mathcal{A}_0$ with $2^{k'} \leq \langle n(\alpha) \rangle < 2^{k'+1}$*

$$
[1,2] - G_{k',\epsilon} \subset I(\alpha, U_k, \epsilon, \delta),
\tag{4.2.14}
$$

where $I(\cdot)$ is defined by (4.1.8). The same conclusion holds when $k' = 0$, $\alpha \in \mathcal{A}_0$.

Proof: Consider first the case $k' \neq 0$. Let $\omega \in [1,2] - \mathcal{O}_{k',\epsilon}$. Take $\ell \in \{1, \ldots, D_\alpha\}$. By (i) of proposition 4.1.1 applied to $(U, U') = (U_k, U_{k'-1})$, there is $\ell' \in \{1, \ldots, D_\alpha\}$ such that

$$
\begin{aligned}
|\lambda_\ell^\alpha(\omega; U_k, \epsilon)| &\geq |\lambda_{\ell'}^\alpha(\omega; U_{k'-1}, \epsilon)| - C_0 \epsilon \|U_k - U_{k'-1}\|_{\mathcal{H}^\sigma} \\
&\geq 2\delta 2^{-k'\zeta} - 2C_0 B_2 \frac{\epsilon^2}{\delta} \frac{2^{-k'\zeta}}{1 - 2^{-\zeta}},
\end{aligned}
\tag{4.2.15}
$$

where the second lower bound follows from the definition (4.2.4) of $\mathcal{O}_{k'}$ and from (4.2.9). Since $\epsilon \leq \delta^2$, we obtain the lower bound

$$
|\lambda_\ell^\alpha(\omega; U_k, \epsilon)| \geq \frac{3}{2} \delta 2^{-k'\zeta}
\tag{4.2.16}
$$

if $\omega \in [1,2] - \mathcal{O}_{k',\epsilon}$ and $\delta \in [0, \delta_0]$ with δ_0 small enough. If $\omega \in \mathcal{O}_{k',\epsilon} - G_{k',\epsilon}$, we take $\tilde{\omega} \in [1,2] - \mathcal{O}_{k',\epsilon}$ with $|\omega - \tilde{\omega}| < \frac{\delta}{8C_0} 2^{-k'(\zeta+2)}$. By (4.1.6), (4.1.7), we know that for any $U \in \mathcal{E}^\sigma(\zeta)$ with $\|U\|_{\mathcal{E}^\sigma(\zeta)} < q$, any $\alpha \in \mathcal{A} - \mathcal{A}_0$, any $\ell \in \{1, \ldots, D_\alpha\}$,

$$
\sup_{\omega' \in [1,2]} |\partial_\omega \lambda_\ell^\alpha(\omega'; U, \epsilon)| \leq C_0 \langle n(\alpha) \rangle^2.
$$

Enlarging C_0, we may assume that this inequality is also valid when $\alpha \in \mathcal{A}_0$. By condition (4.2.11), we may apply it when $U = U_k$. Using (4.2.16), we get since $2^{2k'} \leq \langle n(\alpha)\rangle^2 < 2^{2(k'+1)}$

$$\begin{aligned}|\lambda_\ell^\alpha(\omega; U_k, \epsilon)| &\geq |\lambda_\ell^\alpha(\tilde{\omega}; U_k, \epsilon)| - C_0\langle n(\alpha)\rangle^2 |\omega - \tilde{\omega}|\\ &\geq \delta 2^{-k'\zeta} \geq \delta\langle n(\alpha)\rangle^{-\zeta}.\end{aligned}$$

When $k' = 0$, we argue in the same way, taking in (4.2.15) $U_{k'-1} = 0$. This shows that ω belongs to $I(\alpha, U_k, \epsilon, \delta)$. □

To solve equation (4.2.13), we shall need, in addition to the preceding lemma, estimates for its right hand side. Set

$$\begin{aligned}H_{k+1}(U_k, W_k) = \widetilde{S}_{k+1}(\mathrm{Id} + \epsilon Q(U_k, \omega, \epsilon)^*)R(U_k, \omega, \epsilon)U_k\\ + \widetilde{S}_{k+1}R_1(U_k, \omega, \epsilon)W_k\\ + \widetilde{S}_{k+1}(\mathrm{Id} + \epsilon Q(U_k, \omega, \epsilon)^*)f.\end{aligned} \tag{4.2.17}$$

Lemma 4.2.3 *There is a constant $C > 0$, depending on q in (4.2.11) but independent of k, such that for any $\omega \in [1, 2]$, any $\epsilon \in [0, \delta^2]$, any $\delta \in]0, \delta_0]$*

$$\|H_{k+1}(U_k, W_k)\|_{\mathcal{H}^{s+\zeta}} \leq C[\|U_k(\cdot, \omega, \epsilon)\|_{\mathcal{H}^s} + \|W_k(\cdot, \omega, \epsilon)\|_{\mathcal{H}^s}] + (1 + C\epsilon)\|f\|_{\mathcal{H}^{s+\zeta}}, \tag{4.2.18}$$

$$\begin{aligned}\|\partial_\omega H_{k+1}(U_k, W_k)\|_{\mathcal{H}^{s-2}} \leq C[&\|U_k(\cdot, \omega, \epsilon)\|_{\mathcal{H}^s} + \|\partial_\omega U_k(\cdot, \omega, \epsilon)\|_{\mathcal{H}^{s-\zeta-2}}\\ &+ \|W_k(\cdot, \omega, \epsilon)\|_{\mathcal{H}^s} + \|\partial_\omega W_k(\cdot, \omega, \epsilon)\|_{\mathcal{H}^{s-\zeta-2}} + \epsilon\|f\|_{\mathcal{H}^{s-2}}],\end{aligned} \tag{4.2.19}$$

$$\begin{aligned}\|H_{k+1}(U_k, W_k) - H_k(U_{k-1}, W_{k-1})\|_{\mathcal{H}^{\sigma+\zeta}} \leq C[\|U_k - U_{k-1}\|_{\mathcal{H}^\sigma} + \|W_k - W_{k-1}\|_{\mathcal{H}^\sigma}]\\ + 2^{-k\zeta}[C(\|U_k\|_{\mathcal{H}^{\sigma+\zeta}} + \|W_k\|_{\mathcal{H}^{\sigma+\zeta}}) + (1 + C\epsilon)\|f\|_{\mathcal{H}^{\sigma+2\zeta}}].\end{aligned} \tag{4.2.20}$$

Proof: The operators R and R_1 belong to $\mathcal{R}_2^r(N + 1, \sigma, q)$ with $r = \zeta$. By definition 2.1.3, and because of the assumption (4.2.1) on the indices, they are bounded from $\mathcal{H}^s$ to $\mathcal{H}^{s+\zeta}$. Moreover, $Q(U_k, \omega, \epsilon)^*$ is in $\Psi^0(N, \sigma, q) \otimes \mathcal{M}_2(\mathbb{R})$, so is bounded on any $\mathcal{H}^s$-space by lemma 2.1.2. This gives (4.2.18).

To obtain (4.2.19), one has to study the boundedness properties of

$$\begin{aligned}\partial_\omega[Q(U_k, \omega, \epsilon)] = \partial_U Q(\cdot, \omega, \epsilon) \cdot (\partial_\omega U_k) + \partial_\omega Q(U_k, \omega, \epsilon),\\ \partial_\omega[R(U_k, \omega, \epsilon)] = \partial_U R(\cdot, \omega, \epsilon) \cdot (\partial_\omega U_k) + \partial_\omega R(U_k, \omega, \epsilon),\\ \partial_\omega[R_1(U_k, \omega, \epsilon)] = \partial_U R_1(\cdot, \omega, \epsilon) \cdot (\partial_\omega U_k) + \partial_\omega R_1(U_k, \omega, \epsilon).\end{aligned} \tag{4.2.21}$$

By (2.1.2), inequalities (4.2.1), and the fact that by (4.2.11) $\partial_\omega U_k$ is uniformly bounded in $\mathcal{H}^{s-\zeta-2} \subset \mathcal{H}^\sigma$, we see that the first line in (4.2.21) is a bounded operator on any space $\mathcal{H}^{s'}$. By (2.1.3), and the assumption $s \geq \sigma + \zeta + 2$ in (4.2.1), we see in the same way that the second and third lines in (4.2.21) give bounded operators from $\mathcal{H}^{s-\zeta-2}$ to $\mathcal{H}^{s-2}$ and from $\mathcal{H}^s$ to $\mathcal{H}^{s+\zeta}$. This gives estimate (4.2.19).

To prove (4.2.20), let us write the difference $H_{k+1}(U_k, W_k) - H_k(U_{k-1}, W_{k-1})$ from the following quantities:

$$(4.2.22)\qquad \begin{aligned} &(\widetilde{S}_{k+1} - \widetilde{S}_k)(\mathrm{Id} + \epsilon Q(U_k, \omega, \epsilon)^*)R(U_k, \omega, \epsilon)U_k, \\ &(\widetilde{S}_{k+1} - \widetilde{S}_k)R_1(U_k, \omega, \epsilon)W_k, \\ &(\widetilde{S}_{k+1} - \widetilde{S}_k)(\mathrm{Id} + \epsilon Q(U_k, \omega, \epsilon)^*)f, \end{aligned}$$

$$(4.2.23)\qquad \begin{aligned} &\epsilon\widetilde{S}_k[Q(U_k, \omega, \epsilon)^* - Q(U_{k-1}, \omega, \epsilon)^*]R(U_k, \omega, \epsilon)U_k, \\ &\widetilde{S}_k(\mathrm{Id} + \epsilon Q(U_{k-1}, \omega, \epsilon)^*)[R(U_k, \omega, \epsilon) - R(U_{k-1}, \omega, \epsilon)]U_k, \\ &\widetilde{S}_k[R_1(U_k, \omega, \epsilon) - R_1(U_{k-1}, \omega, \epsilon)]W_k, \\ &\epsilon\widetilde{S}_k[Q(U_k, \omega, \epsilon)^* - Q(U_{k-1}, \omega, \epsilon)^*]f, \end{aligned}$$

$$(4.2.24)\qquad \begin{aligned} &\widetilde{S}_k(\mathrm{Id} + \epsilon Q(U_{k-1}, \omega, \epsilon)^*)R(U_{k-1}, \omega, \epsilon)(U_k - U_{k-1}), \\ &\widetilde{S}_k R_1(U_k, \omega, \epsilon)(W_k - W_{k-1}). \end{aligned}$$

By (4.2.6) and (4.2.9), U_k, W_k stay in a bounded subset of $\mathcal{H}^\sigma$ and R, R_1 act from $H^{\sigma+\zeta}$ to $H^{\sigma+2\zeta}$. Using the cut-off $\widetilde{S}_{k+1} - \widetilde{S}_k$, we see that the $\mathcal{H}^{\sigma+\zeta}$ norm of (4.2.22) is bounded from above by the last term in the right hand side of (4.2.20).

By (2.1.3), the $\mathcal{L}(\mathcal{H}^\sigma, \mathcal{H}^{\sigma+\zeta})$ operator norm of $R(U_k, \omega, \epsilon) - R(U_{k-1}, \omega, \epsilon)$ and of $R_1(U_k, \omega, \epsilon) - R_1(U_{k-1}, \omega, \epsilon)$ is bounded from above by $C\|U_k - U_{k-1}\|_{\mathcal{H}^\sigma}$. By (2.1.2), the $\mathcal{L}(\mathcal{H}^{\sigma+\zeta}, \mathcal{H}^{\sigma+\zeta})$-norm of $Q(U_k, \omega, \epsilon)^* - Q(U_{k-1}, \omega, \epsilon)^*$ is bounded by the same quantity. This shows that the $\mathcal{H}^{\sigma+\zeta}$ norm of (4.2.23) is bounded from above by the right hand side of (4.2.20).

Finally, (4.2.24) is trivially estimated. This concludes the proof. □

End of proof of proposition 4.2.1: We have seen that $\widetilde{\Pi}_\alpha W_{k+1}$ is a solution to equation (4.2.13). Let $k' \in \{1, \ldots, k+1\}$ and $\alpha \in \mathcal{A} - \mathcal{A}_0$ such that $2^{k'} \leq \langle n(\alpha) \rangle < 2^{k'+1}$, or $k' = 0, \alpha \in \mathcal{A}_0$. Let $\omega \in [1, 2] - G_{k',\epsilon}$. By lemma 4.2.2 and proposition 4.1.1, the operator $A_\alpha(\omega; U_k, \epsilon)$ is invertible, ant its inverse satisfies estimates (4.1.9). For such ω, we may write equation (4.2.13)

$$(4.2.25)\qquad \widetilde{\Pi}_\alpha W_{k+1} = \epsilon A_\alpha(\omega; U_k, \epsilon)^{-1}\widetilde{\Pi}_\alpha H_{k+1}(U_k, W_k).$$

Applying estimate (4.1.9), we obtain that for any $k' \in \{1, \ldots, k+1\}$, any $\alpha \in \mathcal{A} - \mathcal{A}_0$ with $2^{k'} \leq \langle n(\alpha) \rangle < 2^{k'+1}$, any $(\omega, \epsilon) \in [1, 2] \times [0, \delta^2] - G_{k'}$ (resp. for any $\alpha \in \mathcal{A}_0$, any $(\omega, \epsilon) \in [1, 2] \times [0, \delta^2] - G_0$)

$$(4.2.26)\qquad \|\widetilde{\Pi}_\alpha W_{k+1}(\cdot, \omega, \epsilon)\|_{\mathcal{H}^s} \leq E_0\frac{\epsilon}{\delta}\|\widetilde{\Pi}_\alpha H_{k+1}(U_k, W_k)(\cdot, \omega, \epsilon)\|_{\mathcal{H}^{s+\zeta}}.$$

In the same way, one gets the estimate

$$(4.2.27)\qquad \begin{aligned} \|\widetilde{\Pi}_\alpha \partial_\omega W_{k+1}(\cdot, \omega, \epsilon)\|_{\mathcal{H}^{s-\zeta-2}} \leq E_0\frac{\epsilon}{\delta}&\|\widetilde{\Pi}_\alpha \partial_\omega H_{k+1}(U_k, W_k)(\cdot, \omega, \epsilon)\|_{\mathcal{H}^{s-2}} \\ &+ E_0\frac{\epsilon}{\delta^2}\|\widetilde{\Pi}_\alpha H_{k+1}(U_k, W_k)(\cdot, \omega, \epsilon)\|_{\mathcal{H}^{s+\zeta}}. \end{aligned}$$

We define $W_{k+1}(t,x,\omega,\epsilon)$ for any value of (ω,ϵ) in $[1,2]\times[0,\delta^2]$ from (4.2.25) setting

$$\begin{aligned}W_{k+1}(t,x,\omega,\epsilon) = \sum_{k'=1}^{k+1} \sum_{\substack{\alpha\in\mathcal{A}-\mathcal{A}_0\\ 2^{k'}\leq\langle n(\alpha)\rangle<2^{k'+1}}} &(1-\psi_{k'}(\omega,\epsilon))\widetilde{\Pi}_\alpha W_{k+1}(t,x,\omega,\epsilon)\\ &+\sum_{\alpha\in\mathcal{A}_0}(1-\psi_0)(\omega,\epsilon)\widetilde{\Pi}_\alpha W_{k+1}(t,x,\omega,\epsilon).\end{aligned}\tag{4.2.28}$$

Note that the right hand side is well defined since (4.2.25) determines $\widetilde{\Pi}_\alpha W_{k+1}(\cdot,\omega,\epsilon)$ on the support of $1-\psi_{k'}$ when (α,k') satisfy the conditions in the summation.

We combine (4.2.28), (4.2.26) and (4.2.18). Taking into account (4.2.6) and (4.2.9), we get

$$\|W_{k+1}(\cdot,\omega,\epsilon)\|_{\mathcal{H}^s} \leq E_0\frac{\epsilon}{\delta}\big[C(B_1+B_2)\frac{\epsilon}{\delta}+\|f\|_{\mathcal{H}^{s+\varsigma}}(1+C\epsilon)\big].\tag{4.2.29}$$

To bound the ∂_ω-derivative, we use that by (4.2.5)

$$\|\partial_\omega\psi_{k'}\widetilde{\Pi}_\alpha W_{k+1}\|_{\mathcal{H}^{s-\varsigma-2}} \leq \frac{C_1}{\delta}\|\widetilde{\Pi}_\alpha W_{k+1}\|_{\mathcal{H}^s}$$

when $2^{k'}\leq\langle n(\alpha)\rangle<2^{k'+1}$, $\alpha\in\mathcal{A}-\mathcal{A}_0$ if $k'\neq 0$, and when $\alpha\in\mathcal{A}_0$ if $k'=0$. We apply this inequality together with (4.2.28), (4.2.27), (4.2.18), (4.2.19) and the uniform bounds (4.2.6), (4.2.9), to get

$$\begin{aligned}\|\partial_\omega W_{k+1}(\cdot,\omega,\epsilon)\|_{\mathcal{H}^{s-\varsigma-2}} \leq E_0\frac{\epsilon}{\delta}&\big[C(B_1+B_2)\frac{\epsilon}{\delta^2}+C\epsilon\|f\|_{\mathcal{H}^{s-2}}\big]\\ &+E_0\frac{\epsilon}{\delta^2}\big[C(B_1+B_2)\frac{\epsilon}{\delta}+(1+C\epsilon)\|f\|_{\mathcal{H}^{s+\varsigma}}\big]\\ &+E_0C_1\frac{\epsilon}{\delta^2}\big[C(B_1+B_2)\frac{\epsilon}{\delta}+(1+C\epsilon)\|f\|_{\mathcal{H}^{s+\varsigma}}\big].\end{aligned}\tag{4.2.30}$$

In (4.2.29), (4.2.30), C depends on the *a priori* bound given by (4.2.11), while E_0, C_1 are uniform constants. Consequently, if we take B_1 large enough relatively to $\|f\|_{\mathcal{H}^{s+\varsigma}}$, E_0, C_1 and then $\epsilon\leq\delta^2\leq\delta_0^2$, with δ_0 small enough, we deduce from (4.2.29), (4.2.30) that (4.2.6) holds at rank $k+1$. The second estimate (4.2.9) at rank $k+1$ follows, with for instance $B_2=2B_1$, if δ_0 is small enough. We are left with establishing the first estimate (4.2.9) at rank $k+1$ and (4.2.10).

First let us bound $W_{k+1}-W_k$. By (4.2.25), for $k'\in\{1,\dots,k\}$, $(\omega,\epsilon)\in[1,2]\times[0,\delta^2]-G_{k'}$, $\alpha\in\mathcal{A}-\mathcal{A}_0$, $2^{k'}\leq\langle n(\alpha)\rangle<2^{k'+1}$, or for $(\omega,\epsilon)\in[1,2]\times[0,\delta^2]-G_0$ and $\alpha\in\mathcal{A}_0$,

$$\begin{aligned}(\widetilde{L}_\omega+\epsilon V_{\mathrm{D}}(U_k,\omega,\epsilon))\widetilde{\Pi}_\alpha W_{k+1} &= \epsilon\widetilde{\Pi}_\alpha H_{k+1}(U_k,W_k)\\ (\widetilde{L}_\omega+\epsilon V_{\mathrm{D}}(U_{k-1},\omega,\epsilon))\widetilde{\Pi}_\alpha W_k &= \epsilon\widetilde{\Pi}_\alpha H_k(U_{k-1},W_{k-1})\end{aligned}$$

whence the equation

$$\begin{aligned}(\widetilde{L}_\omega+\epsilon V_{\mathrm{D}}(U_k,\omega,\epsilon))\widetilde{\Pi}_\alpha(W_{k+1}-W_k) = \epsilon\widetilde{\Pi}_\alpha&[V_{\mathrm{D}}(U_{k-1},\omega,\epsilon)-V_{\mathrm{D}}(U_k,\omega,\epsilon)]W_k\\ &+\epsilon\widetilde{\Pi}_\alpha[H_{k+1}(U_k,W_k)-H_k(U_{k-1},W_{k-1})].\end{aligned}\tag{4.2.31}$$

We make act $\mathcal{A}_\alpha(\omega; U_k, \epsilon)^{-1}$ on both sides as in (4.2.25). Applying inequality (4.1.9) we get

$$
\begin{aligned}
\|\widetilde{\Pi}_\alpha(W_{k+1}-W_k)\|_{\mathcal{H}^\sigma} \leq \frac{E_0\epsilon}{\delta}[&\|\widetilde{\Pi}_\alpha[V_{\mathrm{D}}(U_{k-1},\omega,\epsilon)-V_{\mathrm{D}}(U_k,\omega,\epsilon)]W_k\|_{\mathcal{H}^{\sigma+\zeta}} \\
&+\|\widetilde{\Pi}_\alpha[H_{k+1}(U_k,W_k)-H_k(U_{k-1},W_{k-1})]\|_{\mathcal{H}^{\sigma+\zeta}}].
\end{aligned}
\tag{4.2.32}
$$

This estimate holds outside $G_{k'}$ (resp. G_0) when $k' \neq 0$, $\alpha \in \mathcal{A}-\mathcal{A}_0$, $2^{k'} \leq \langle n(\alpha)\rangle < 2^{k'+1}$ (resp. $\alpha \in \mathcal{A}_0$). By (4.2.28), we may write

$$
\begin{aligned}
(W_{k+1}-W_k)(t,x,\omega,\epsilon) = &\sum_{\alpha\in\mathcal{A}_0}(1-\psi_0)\widetilde{\Pi}_\alpha(W_{k+1}-W_k) \\
&+\sum_{k'=1}^{k}\sum_{\substack{\alpha\in\mathcal{A}-\mathcal{A}_0 \\ 2^{k'}\leq\langle n(\alpha)\rangle<2^{k'+1}}}(1-\psi_{k'})\widetilde{\Pi}_\alpha(W_{k+1}-W_k) \\
&+\sum_{\substack{\alpha\in\mathcal{A}-\mathcal{A}_0 \\ 2^{k+1}\leq\langle n(\alpha)\rangle<2^{k+2}}}(1-\psi_{k+1})\widetilde{\Pi}_\alpha W_{k+1}.
\end{aligned}
\tag{4.2.33}
$$

The $\mathcal{H}^\sigma$ norm of the last term is bounded by $C_2 2^{-k(s-\sigma)}\|W_{k+1}\|_{\mathcal{H}^s} \leq C_2 B_1 \frac{\epsilon}{\delta} 2^{-k(s-\sigma)}$ by (4.2.6), for some universal constant C_2. The $\mathcal{H}^\sigma$-norm of the k'-sum in (4.2.33) may be estimated using (4.2.32), (4.2.20) and the bound

$$
\|(V_{\mathrm{D}}(U_{k-1},\omega,\epsilon)-V_{\mathrm{D}}(U_k,\omega,\epsilon))W_k\|_{\mathcal{H}^{\sigma+\zeta}} \leq C\|U_k-U_{k-1}\|_{\mathcal{H}^\sigma}\|W_k\|_{\mathcal{H}^s}
$$

which follows from (2.1.2), and where we used $s \geq \sigma+\zeta$. Using the induction hypothesis (4.2.9), (4.2.10), we get

$$
\begin{aligned}
\|W_{k+1}-W_k\|_{\mathcal{H}^\sigma} \leq E_0\frac{\epsilon}{\delta}[&CB_1\frac{\epsilon}{\delta}2B_2\frac{\epsilon}{\delta}2^{-k\zeta}+3CB_2\frac{\epsilon}{\delta}2^{-k\zeta}+C2^{-k\zeta}(B_1+B_2)\frac{\epsilon}{\delta} \\
&+(1+C\epsilon)\|f\|_{\mathcal{H}^{\sigma+2\zeta}}2^{-k\zeta}] \\
&+C_2B_1\frac{\epsilon}{\delta}2^{-k(s-\sigma)}.
\end{aligned}
\tag{4.2.34}
$$

Since $s \geq \sigma+\zeta$, we may take B_1 large enough relatively to E_0, $\|f\|_{\mathcal{H}^{s+\zeta}}$, and B_2 large enough relatively to C_2, B_1, and $\frac{\epsilon}{\delta} \leq \delta \leq \delta_0$ small enough, so that (4.2.34) is smaller than $B_2\frac{\epsilon}{\delta}2^{-(k+1)\zeta}$, whence (4.2.10) at rank $k+1$. Writing

$$
U_{k+1}-U_k = (\mathrm{Id}+\epsilon Q(U_k,\omega,\epsilon))(W_{k+1}-W_k)+\epsilon(Q(U_k,\omega,\epsilon)-Q(U_{k-1},\omega,\epsilon))W_k
$$

we deduce from that the first inequality (4.2.9) at rank $k+1$, for small enough ϵ. This concludes the proof of the proposition. □

Proof of theorem 1.1.1: By (4.2.9), the series $\sum(U_k-U_{k-1})$ converges in $\mathcal{H}^\sigma(\mathbb{S}^1\times\mathbb{T}^d;\mathbb{C}^2)$ and its sum U satisfies $U \in \mathcal{H}^s(\mathbb{S}^1\times\mathbb{T}^d;\mathbb{C}^2)$ with

$$
\|U(\cdot,\omega,\epsilon)\|_{\mathcal{H}^s}+\delta\|\partial_\omega U(\cdot,\omega,\epsilon)\|_{\mathcal{H}^{s-\zeta-2}} \leq B_2\frac{\epsilon}{\delta}.
$$

We have to check that U gives a solution to our problem outside a set of parameters of small measure. Let $(\omega,\epsilon) \in [1,2]\times[0,\delta^2]-\bigcup_{k'=0}^{+\infty}\mathcal{O}_{k'}$, $\delta \in]0,\delta_0]$. Then equation (4.2.12) is satisfied for any k. We make $k \to +\infty$. Since we have uniform $\mathcal{H}^s$ bounds for U_k, W_k and $\mathcal{H}^\sigma$ convergence for these quantities, the limit U satisfies

$$(\widetilde{L}_\omega + \epsilon V(U, \omega, \epsilon))U = \epsilon R(U, \omega, \epsilon)U + \epsilon f$$

that is equation (2.3.15). We have seen that this equation is equivalent to (2.3.14), which is, by proposition 2.3.1, the same as (2.2.13). Since proposition 2.2.4 shows that, up to a change of notations, this equation is equivalent to the formulation (2.2.6) of equation (1.1.3), we obtain a solution satisfying the requirements of theorem 1.1.1. We still have to check that (1.1.5) holds with $\mathcal{O} = \bigcup_{k'=0}^{+\infty} \mathcal{O}_{k'}$. According to (4.2.2), the set $\mathcal{O}_0$ is included in the set of those (ω, ϵ) such that there are (j, n) in a given finite subset of $\mathbb{Z}^2$ such that $|j\omega + |n|^2 + \mu| < 2\delta$. The ω-measure of this set is $O(\delta)$, $\delta \to 0$ (Note that since $\mu \notin \mathbb{Z}_-$, we may always assume $j \neq 0$). For $k' > 0$, $\mathcal{O}_{k'}$ is the union for $\alpha \in \mathcal{A} - \mathcal{A}_0$ with $2^{k'-1} \leq \langle n(\alpha) \rangle < 2^{k'}$ and $\ell \in \{1, \dots, D_\alpha\}$ of the set of those (ω, ϵ) satisfying

$$|\lambda_\alpha^\ell(\omega; U_{k'-1}, \epsilon)| < 2\delta 2^{-k'\zeta}.$$

By (4.1.6), (4.1.7) the ω-measure of each of these sets in bounded by $C\langle n(\alpha) \rangle^{-2} \delta 2^{-k'\zeta} \leq C 2^{-(k'+2)\zeta} \delta$. Since $D_\alpha \leq C_1 2^{k'(\beta d + 2)}$ by (1.2.4), (1.2.6), we obtain for the measure of the ϵ-section of $\mathcal{O}$ the bound

$$C \sum_{k'=0}^{+\infty} 2^{-(k'+2)\zeta + k'(\beta d + 2) + k'd} \delta.$$

If we take $\zeta > (\beta + 1)d + 2$, we obtain the wanted $O(\delta)$ bound. This concludes the proof. $\square$

A Appendix

We gather here some elementary results used throughout the paper.

Lemma A.1 *Let $s > \frac{d}{2} + 1$. Then $\widetilde{\mathcal{H}}^s(\mathbb{S}^1 \times \mathbb{T}^d; \mathbb{C}) \subset L^\infty$. Moreover, if F is a smooth function on $\mathbb{S}^1 \times \mathbb{T}^d \times \mathbb{C}$, satisfying $F(t, x, 0) \equiv 0$, there is some continuous function $\tau \to C(\tau)$ such that for any $u \in \widetilde{\mathcal{H}}^s$, $F(\cdot, u) \in \widetilde{\mathcal{H}}^s$ with the estimate $\|F(\cdot, u)\|_{\widetilde{\mathcal{H}}^s} \leq C(\|u\|_{L^\infty}) \|u\|_{\widetilde{\mathcal{H}}^s}$.*

Proof: Let $\varphi \in C_0^\infty(]0, +\infty[)$, $\varphi \geq 0$, $\varphi \equiv 1$ on $[1, 2]$ be such that $\sum_{\ell=-\infty}^{+\infty} \varphi(2^{-\ell}\lambda) \equiv 1$ for $\lambda \in \mathbb{R}_+^*$, and define $\psi(\lambda) = \sum_{-\infty}^{0} \varphi(2^{-\ell}\lambda)$. Consider for $(j, n) \in \mathbb{Z} \times \mathbb{Z}^d$

$$\begin{aligned} \Phi_k(j, n) &= \varphi(2^{-2k}(j^2 + |n|^4)^{1/2}), \ k \geq 1 \\ \Phi_0(j, n) &= \psi((j^2 + |n|^4)^{1/2}) \end{aligned} \tag{A.1}$$

Define for $u \in \widetilde{\mathcal{H}}^0$, $k \in \mathbb{N}$

$$\begin{aligned} \Delta_k u &= \sum_{j,n} \Phi_k(j, n) \hat{u}(j, n) \frac{e^{i(tj + k \cdot n)}}{(2\pi)^{(d+1)/2}} \\ K_k(t, x) &= \frac{1}{(2\pi)^{d+1}} \sum_{j,n} \Phi_k(j, n) e^{i(tj + k \cdot n)}. \end{aligned} \tag{A.2}$$

Then for any $N \in \mathbb{N}$

$$|K_k(t,x)| \leq C_N 2^{2k\left(1+\frac{d}{2}\right)}(1 + 2^{2k}|e^{it} - 1| + 2^k|e^{ix} - 1|)^{-N} \tag{A.3}$$

and $u \in \widetilde{\mathcal{H}}^s$ if and only if $(2^{ks}\|\Delta_k u\|_{L^2})_k$ is in ℓ^2.

The first statement of the lemma follows from the inequality $\|\Delta_k u\|_{L^\infty} \leq C2^{k\left(1+\frac{d}{2}\right)}\|\Delta_k u\|_{L^2}$, which is a consequence of (A.3) (for the kernel corresponding to an enlarged Φ_k). To get the second statement, we consider first the case of a function F that does not depend on (t,x). We set $S_k = \sum_{k' \leq k-1} \Delta_{k'}$ when $k \geq 1$, $S_0 = 0$ and write

$$F(u) = \sum_{k=0}^{+\infty}(F(S_{k+1}u) - F(S_k u)) = \sum_{k=0}^{+\infty} m_k(u)\Delta_k u$$

where $m_k(u) = \int_0^1 F'(S_k u + \tau\Delta_k u)\, d\tau$. It follows from the definition of S_k that this operator is given by a convolution kernel obeying the same estimates as in (A.3). Consequently, for any $(\alpha, \beta) \in \mathbb{N} \times \mathbb{N}^d$,

$$\|\partial_t^\alpha \partial_x^\beta m_k(u)\|_{L^\infty} \leq C2^{2k\alpha + k|\beta|} \tag{A.4}$$

with a constant depending only on $\|u\|_{L^\infty}$. One writes for some $N_0 \in \mathbb{N}$ to be chosen

$$\Delta_j[F(u)] = \sum_{k=0}^{j-1-N_0} \Delta_j[m_k(u)\Delta_k u] + \sum_{k=j-N_0}^{+\infty} \Delta_j[m_k(u)\Delta_k u]. \tag{A.5}$$

The L^2-norm of the second sum is bounded by $Cc_j 2^{-js}\|u\|_{\widetilde{\mathcal{H}}^s}$ for some sequence $(c_j)_j$ in the unit ball of ℓ^2, and some C depending only on $\|u\|_{L^\infty}$. If N_0 is fixed large enough, because of the support properties of the Fourier transforms,

$$\Delta_j[m_k(u)\Delta_k u] = \Delta_j[[(\mathrm{Id} - S_{j-N_0})m_k(u)]\Delta_k u]$$

when $k \leq j - 1 - N_0$. We estimate the L^2-norm of this quantity by

$$\|(\mathrm{Id} - S_{j-N_0})m_k\|_{L^\infty}\|\Delta_k u\|_{L^2} \tag{A.6}$$

and use that for any N $\|(\mathrm{Id} - S_{j-N_0})m_k\|_{L^\infty} \leq C_N 2^{-4jN}\|P^N m_k\|_{L^\infty}$ where $P = \partial_t^2 + \Delta^2 + 1$. It follows from (A.4) that (A.6) is bounded from above by $C_N 2^{-4(j-k)N}\|\Delta_k u\|_{L^2}$, from which we deduce that the L^2-norm of the first sum in (A.5) is also smaller than $C2^{-js}c_j\|u\|_{\widetilde{\mathcal{H}}^s}$. This concludes the proof for functions F independent of (t,x). In the general case, we note that since u is bounded, we may always assume that F is compactly supported, and we write

$$F(t,x,u) = \frac{1}{2\pi}\int_{\mathbb{R}} F_1(u,\theta)b(t,x,\theta)\, d\theta$$

where $F_1(u,\theta) = e^{iu\theta} - 1$ and $b(t,x,\theta)$ is the Fourier transform of $u \to F(t,x,u)$. Then it follows from the above proof that $F_1(u,\theta)$ is in $\widetilde{\mathcal{H}}^s$ with a bound $\|F_1(u,\theta)\|_{\widetilde{\mathcal{H}}^s} \leq C\langle\theta\rangle^{N(s)}$, for some exponent $N(s)$. Moreover, for any N, $\|b(\cdot,\theta)\|_{\widetilde{\mathcal{H}}^s} \leq C_N\langle\theta\rangle^{-N}$. We get the conclusion by superposition. □

Corollary A.2 *Let $F : \mathbb{S}^1 \times \mathbb{T}^d \times \mathbb{C} \to \mathbb{C}$ be a smooth function with $F(t,x,0) \equiv 0$. Then for any $\sigma > \frac{d}{2}+1$, $u \to F(\cdot,u)$ is a smooth map from $\widetilde{\mathcal{H}}^\sigma$ to itself.*

Proof: We write

$$F(t,x,u+h) - F(t,x,u) - \partial_u F(t,x,u)h = \int_0^1 \int_0^1 (D^2F)(t,x,u+\tau_1\tau_2 h)\tau_1 \cdot h^2 \, d\tau_1 d\tau_2$$

and we apply the lemma to $D^2F(t,x,u) - D^2F(t,x,0)$. □

Lemma A.3 • *Let $s > \frac{d}{2}+1$. If $u \in \widetilde{\mathcal{H}}^s$ and $v \in \widetilde{\mathcal{H}}^{\sigma'}$ for some $\sigma' \in [-s,s]$, then $uv \in \widetilde{\mathcal{H}}^{\sigma'}$.*

• *For any $\sigma \in \mathbb{R}$, $\sigma_0 > \frac{d}{2}+1$, $\widetilde{\mathcal{H}}^\sigma \cdot \widetilde{\mathcal{H}}^{-\sigma} \subset \widetilde{\mathcal{H}}^{-\max(\sigma,\sigma_0)}$.*

References

[1] D. Bambusi and S. Paleari: *Families of periodic solutions of resonant PDEs*, J. Nonlinear Sci. 11 (2001), no. 1, 69–87.

[2] D. Bambusi and S. Paleari: *Families of periodic orbits for some PDE's in higher dimensions*, Commun. Pure Appl. Anal. 1 (2002), no. 2, 269–279.

[3] M. Berti: *Nonlinear oscillations of Hamiltonian PDEs*, Progress in Nonlinear Differential Equations and Their Applications, 74, Birkhäuser, (2007).

[4] M. Berti and P. Bolle: *Cantor families of periodic solutions for completely resonant nonlinear wave equations*, Duke Math. J. 134 (2006), no. 2, 359–419.

[5] M. Berti and P. Bolle: *Sobolev periodic solutions of nonlinear wave equations in higher spatial dimensions*, Arch. Rational Mech. Anal. (2009), to appear.

[6] M. Berti, P. Bolle and G. Procesi: *An abstract Nash-Moser Theorem with parameters and applications to PDEs*, preprint, (2009).

[7] M. Berti and G. Procesi: *Nonlinear wave and Schrödinger equations on compact Lie groups and homogeneous spaces*, preprint, (2009).

[8] J.-M. Bony: *Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires*, Ann. Sci. École Norm. Sup. (4) 14 (1981), no. 2, 209–246.

[9] J. Bourgain: *Construction of quasi-periodic solutions for Hamiltonian perturbations of linear equations and applications to nonlinear PDE*, Internat. Math. Res. Notices (1994), no. 11, 475–497.

[10] J. Bourgain: *Construction of periodic solutions of nonlinear wave equations in higher dimension*, Geom. Funct. Anal. 5 (1995), no. 4, 629–639.

[11] J. Bourgain: *Quasi-periodic solutions of Hamiltonian perturbations of 2D linear Schrödinger equations*, Ann. of Math. (2) 148 (1998), no. 2, 363–439.

[12] J. Bourgain: *On growth of Sobolev norms in linear Schrödinger equations with smooth time dependent potential*, J. Anal. Math. 77 (1999), 315–348.

[13] J. Bourgain: *Green's function estimates for lattice Schrödinger operators and applications*, Annals of Mathematics Studies, 158, Princeton University Press, 2005.

[14] W. Craig: *Problèmes de petits diviseurs dans les équations aux dérivées partielles*, Panoramas et Synthèse, 9. Société Mathématique de France, Paris, 2000. viii+120 pp.

[15] W. Craig and E. Wayne: *Newton's method and periodic solutions of nonlinear wave equations*, Comm. Pure Appl. Math. 46 (1993), no. 11, 1409–1498.

[16] W. Craig and E. Wayne: *Periodic solutions of nonlinear Schrödinger equations and the Nash-Moser method*, Hamiltonian mechanics (Toruń, 1993), 103–122, NATO Adv. Sci. Inst. Ser. B Phys., 331, Plenum, New York, 1994.

[17] J.-M. Delort: *Growth of Sobolev norms of solutions of linear Schrödinger equations on some compact manifolds*, preprint, (2009).

[18] H. Eliasson and S. Kuksin: *KAM for the non-linear Schrödinger equation*, preprint, (2008), to appear, Annals of Mathematics.

[19] G. Gentile and M. Procesi: *Periodic solutions for a class of nonlinear partial differential equations in higher dimension*, Comm. Math. Phys. 289 (2009), no. 3, 863–906.

[20] T. Kato: *Perturbation theory for linear operators*, Classics in Mathematics. Springer-Verlag, Berlin, 1995. xxii+619.

[21] S. Kuksin: *Nearly integrable infinite-dimensional Hamiltonian systems*, Lecture Notes in Mathematics, 1556. Springer-Verlag, Berlin, 1993. xxviii+101 pp.

[22] S. Kuksin: *Analysis of Hamiltonian PDEs*, Oxford Lecture Series in Mathematics and its Applications, 19. Oxford University Press, Oxford, 2000. xii+212 pp.

[23] S. Kuksin and J. Pöschel: *Invariant Cantor manifolds of quasi-periodic oscillations for a nonlinear Schrödinger equation*, Ann. of Math. (2) 143 (1996), no. 1, 149–179.

[24] E. Wayne: *Periodic and quasi-periodic solutions of nonlinear wave equations via KAM theory*, Comm. Math. Phys. 127 (1990), no. 3, 479–528.